Global geometry and topology of spacelike stationary surfaces in $\mathbb{R}^4_1$

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Abstract

For spacelike stationary (i.e. zero mean curvature) surfaces in 4-dimensional Lorentz space one can naturally introduce two Gauss maps and Weierstrass representation. In this paper we investigate their global geometry systematically. The total Gaussian curvature is related with the surface topology as well as the indices of the so-called good singular ends by a generalized Jorge-Meeks formula. On the other hand, as shown by a family of counter-examples to Osserman's theorem, finite total curvature no longer implies that Gauss maps extend to the ends. Interesting examples include the generalization of the classical catenoids, helicoids, the Enneper surface, and Jorge-Meeks' k-noids. Any of them could be embedded in $\mathbb{R}^4_1$, showing a sharp contrast with the case of $\mathbb{R}^3$.

Keywords: stationary surface, minimal surface, Weierstrass representation, Gauss maps, total curvature, Gauss-Bonnet formula, index of singularities, singular end, essential singularity, embedding

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Contents

1 Introduction

Zero mean curvature spacelike surfaces in 4-dimensional Lorentz space $\mathbb{R}^4_1$ include classical minimal surfaces in $\mathbb{R}^3$ and maximal surfaces in $\mathbb{R}^3_1$ as special cases. They are no longer local minimizer or maximizer of the area functional. Hence we call them *stationary surfaces* which are always assumed to be spacelike in this paper.

We got interested in this topic when studying several surface classes arising from variational problems in Möbius geometry and Laguerre geometry [23], [31]. After reducing our original problems to stationary surfaces in $\mathbb{R}^4_1$, we searched the literature and found very few papers on this general case, which contrasted sharply to the rich theory and deep results on minimal surfaces in $\mathbb{R}^3$ (see the recent survey [26]) and maximal surfaces in $\mathbb{R}^3_1$ [30].

This situation motivated us to extend the general theory about the global geometry and topology of minimal surfaces in $\mathbb{R}^3$ to these stationary surfaces in $\mathbb{R}^4_1$. As a preparation, Section 2 introduces the basic invariants, equations, and the Weierstrass type representation formula in terms of two meromorphic functions ϕ, ψ (corresponding to two Gauss maps, namely the two lightlike normal directions) and a holomorphic 1-form $\mathrm{d}h$ (the height differential). Here we see that the local geometry of stationary surfaces are quite similar to usual minimal surfaces in $\mathbb{R}^3$.

But stationary surfaces have quite different global geometry. The classical Osserman's theorem says that a complete minimal surface in $\mathbb{R}^3$ with finite total Gaussian curvature always has well-defined limit for the Gauss map at each end. In contrast with this, in Section 3 we construct complete stationary surfaces with finite total curvature, whose Gauss maps ϕ, ψ have some essential singularities at the ends (hence the Weierstrass data could not extend analytically to the whole compactified Riemann surface).

Secondly, the behavior of ends is more complicated. There exists the so-called *singular end* where the limits of the two lightlike normal directions coincide. In terms of the Weierstrass data ϕ, ψ, at the end we have $\phi = \bar{\psi}$. When they take this limit value with the same multiplicity, the total Gaussian curvature will diverge. Such an end is called *bad singular end*. In the other case we define index for *good singular end*. These are discussed in Section 4. We then derive a Gauss-Bonnet type theorem in Section 5 for algebraic stationary surfaces (i.e. the Weierstrass data extend to meromorphic functions/forms on compact Riemann surfaces) without bad singular ends. This generalizes the Jorge-Meeks formula in $\mathbb{R}^3$ and provides a cornerstone for constructing various examples with given global behavior.

From Section 6 to Section 8 we generalize classical examples like catenoids, helicoids, the Enneper surface, and Jorge-Meeks' k-noids. In particular, the generalized Enneper surfaces and k-noids could be embedded in $\mathbb{R}^4_1$. Thus many uniqueness theorems for completely embedded minimal surfaces in $\mathbb{R}^3$ no longer hold true, and embeddedness is no longer such a strong restriction on the global geometry and topology of such surfaces in $\mathbb{R}^4_1$. We just mention two most famous results among them.

Theorem (Lopez-Ros theorem [21]): Complete, embedded minimal surfaces in $\mathbb{R}^3$ with genus 0 and finite total curvature are planes and catenoids.

Theorem (Meeks and Rosenberg [24]): Helicoids are the only non-flat, properly embedded, simply connected, complete minimal surfaces in $\mathbb{R}^3$.

The reader might think that in such a codimension-2 case it is easy to deform any surface and avoid self-intersection. But one should keep in mind of the other restrictions such as being spacelike and complete. Besides that, transversal intersection is still possible which could not be eliminated by small perturbation. Thus the existence of many complete and embedded examples in $\mathbb{R}^4_1$ is a non-trivial fact. It is a very interesting question whether one can establish some similar uniqueness theorem under the assumption of embedding in this new context. See discussions in Section 9 among other open problems.

We would like to mention some previous work on stationary surfaces which are also important motivation to our work. Estudillo and Romero [9] considered the exceptional value problem for the normal directions and established certain Bernstein type theorems for complete stationary surfaces in $\mathbb{R}^n_1$. Alías and Palmer [2] dealt with curvature properties of such surfaces in $\mathbb{R}^4_1$. The Weierstrass type representation in $\mathbb{R}^4_1$ should also be known already. It seems quite natural to extend from minimal surfaces in $\mathbb{R}^3$ to our case by using the classical method systematically. What puzzled and surprised us is that nobody did this before (to the best of our knowledge). The main reasons might be as follows.

First, according to our observation, there do exist richer phenomenon and new difficulties in $\mathbb{R}^4_1$. The Osserman's theorem fails, and the singular ends as well as the problem of solving the equation $\phi = \bar{\psi}$ are new challenges not so easy to overcome.

Second, as pointed out at the beginning, the stability property of stationary surfaces in $\mathbb{R}^4_1$ is bad. People might not have great interest in considering a surface class without much physical significance. (But under suitable restrictions on the allowed variations, there are still stability results. See Palmer's work [1],[29].)

Last but not least, the embedding problem gets harder, and the 4-dimensional case loses the beautiful geometric intuition which is always so appealing in the 3-space. For the purpose of visualization, one may take a slice (i.e. the intersection of $\vec{x}(M)$ with a 3-space $x_4 = 0$ or $x_3 = 0$), or a projection to such 3-spaces. But neither of them is satisfactory.

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2 Weierstrass Representation

Let $\vec{x} : M^2 \to \mathbb{R}^4_1$ be an oriented complete spacelike surface in 4-dimensional Lorentz space with local complex coordinate $z = u + \mathrm{i}v$ and zero mean curvature $\vec{H} = 0$. The Lorentz inner product $\langle \cdot, \cdot \rangle$ is given by

$$\langle \vec{x}, \vec{x} \rangle = x_1^2 + x_2^2 + x_3^2 - x_4^2.$$

The induced Riemannian metric is $\mathrm{d}s^2 = \mathrm{e}^{2\omega}|\mathrm{d}z|^2$ on M. Hence

$$\langle \vec{x}_z, \vec{x}_z \rangle = 0, \quad \langle \vec{x}_z, \vec{x}_{\bar{z}} \rangle = \frac{1}{2}\mathrm{e}^{2\omega}.$$

Choose null vectors $\vec{y}, \vec{y}^*$ in the normal plane $\top^{\perp}_{\vec{x}}M$ at each point such that

$$\langle\vec{y},\vec{y}\rangle = \langle\vec{y}^*,\vec{y}^*\rangle = 0, \quad \langle\vec{y},\vec{y}^*\rangle = 1, \quad \det\{\vec{x}_u,\vec{x}_v,\vec{y},\vec{y}^*\} > 0 \ .$$

Such frames $\{\vec{y},\ \vec{y}^*\}$ are determined up to scaling

$$\{\vec{y},\ \vec{y}^*\} \to \{\lambda\vec{y},\ \lambda^{-1}\vec{y}^*\} \tag{1}$$

for some non-zero real-valued function λ. After projection, we obtain two well-defined maps (independent to the scaling (1))

$$[\vec{y}],\ [\vec{y}^*] : M \to S^2 \cong \{[\vec{v}] \in \mathbb{R}P^3 | \langle\vec{v},\vec{v}\rangle = 0\}.$$

The target space is usually called the projective light-cone, which is well-known to be homeomorphic to the 2-sphere. By analogy to $\mathbb{R}^3$, we call them *Gauss maps* of the spacelike surface $\vec{x}$ in $\mathbb{R}^4_1$.

The structure equations is derived under the (complex) moving frame $\{\vec{x}_z, \vec{x}_{\bar{z}}, \vec{y}, \vec{y}^*\}$:

$$\vec{x}_{zz} = 2\omega_z\ \vec{x}_z + \Omega^*\ \vec{y} + \Omega\ \vec{y}^*, \tag{2}$$

$$\vec{x}_{z\bar{z}} = 0, \tag{3}$$

$$\vec{y}_z = \alpha\ \vec{y} - 2\mathrm{e}^{-2\omega}\Omega\ \vec{x}_{\bar{z}}, \tag{4}$$

$$\vec{y}^*_z = -\alpha\ \vec{y}^* - 2\mathrm{e}^{-2\omega}\Omega^*\ \vec{x}_{\bar{z}}. \tag{5}$$

Here $\Omega \triangleq \langle\vec{x}_{zz},\vec{y}\rangle, \Omega^* \triangleq \langle\vec{x}_{zz},\vec{y}^*\rangle$ are only quasi-invariants (since they are defined only up to scaling (1)) whose geometric meaning is similar to the usual Hopf differential. $\alpha\mathrm{d}z \triangleq \langle\vec{y}_z,\vec{y}^*\rangle\mathrm{d}z$ gives the connection 1-form of the normal bundle.

Note that equation (3) means $\vec{x}$ is still a vector-valued harmonic function, which is equivalent to the condition $\vec{H} = 0$ (zero mean curvature).

The integrability equations are:

$$K = -4\mathrm{e}^{-2\omega}\omega_{z\bar{z}} = -4\mathrm{e}^{-4\omega}(\Omega\overline{\Omega^*} + \bar{\Omega}\Omega^*)\ , \tag{6}$$

$$\Omega_{\bar{z}} - \bar{\alpha}\Omega = \Omega^*_{\bar{z}} + \bar{\alpha}\Omega^* = 0\ , \tag{7}$$

$$K^{\perp} = -2\mathrm{i}\cdot\mathrm{e}^{-2\omega}(\alpha_{\bar{z}} - \bar{\alpha}_z) = -4\mathrm{i}\cdot\mathrm{e}^{-4\omega}(\Omega\overline{\Omega^*} - \bar{\Omega}\Omega^*)\ . \tag{8}$$

Here $K, K^{\perp}$ are the Gaussian curvature and the normal curvature, respectively. The equations (6) and (8) may be combined to form a single formula:

$$-K + \mathrm{i}K^{\perp} = 8\mathrm{e}^{-4\omega}\Omega\overline{\Omega^*}\ . \tag{9}$$

Together with (7), it follows that

$$\Delta\ln(-K + \mathrm{i}K^{\perp}) = -4K + 2K^{\perp}\mathrm{i}\ , \tag{10}$$

where $\Delta \triangleq 4\mathrm{e}^{-2\omega}\frac{\partial^2}{\partial z\partial\bar{z}}$ is the usual Laplacian operator with respect to $\mathrm{d}s^2$. (Similar formulas have appeared in [2].)

The assumption of zero mean curvature implies that the two Gauss maps $[\vec{y}],\ [\vec{y}^*] : M \to S^2$ are conformal, i.e.

$$\langle\vec{y}_z,\vec{y}_z\rangle = \langle\vec{y}^*_z,\vec{y}^*_z\rangle = 0\ ,$$

by (4)(5). In particular, they induce opposite orientations on the target space S^2 according to the observation in [31]. Assume that $[\vec{y}]$ is given by a meromorphic

function $\phi = \phi^1 + \mathrm{i}\phi^2 : M \to \mathbb{C} \cup \{\infty\}$, and $[\vec{y}^*]$ is given by the complex conjugation of a meromorphic function, i.e. $\bar{\psi} = \psi^1 - \mathrm{i}\psi^2 : M \to \mathbb{C} \cup \{\infty\}$.

Since $[\vec{y}] \neq [\vec{y}^*]$ at any point, we have $\phi \neq \bar{\psi}$ over M and they do not have poles at the same regular point. Denote

$$\phi - \bar{\psi} = \rho \mathrm{e}^{\mathrm{i}\theta}, \;\; \rho > 0 \; . \tag{11}$$

Then we can write

$$\vec{y} = \frac{\sqrt{2}}{\rho}\left(\phi^1, \phi^2, \frac{1-|\phi|^2}{2}, \frac{1+|\phi|^2}{2}\right), \tag{12}$$

$$\vec{y}^* = \frac{\sqrt{2}}{\rho}\left(\psi^1, -\psi^2, \frac{1-|\psi|^2}{2}, \frac{1+|\psi|^2}{2}\right) \; . \tag{13}$$

Since $\phi_{\bar{z}} = \psi_{\bar{z}} = 0$, by a direct calculation we have

$$\alpha = \langle \vec{y}_z, \vec{y}^* \rangle = \mathrm{i}\theta_z, \tag{14}$$

$$\vec{y}_z - \alpha\vec{y} = -\frac{1}{\sqrt{2}\rho^2}\mathrm{e}^{-\mathrm{i}\theta}\phi_z\Big(\bar{\phi}+\bar{\psi}, i(\bar{\phi}-\bar{\psi}), 1-\bar{\phi}\bar{\psi}, 1+\bar{\phi}\bar{\psi}\Big) \; , \tag{15}$$

$$\vec{y}^*_z + \alpha\vec{y}^* = -\frac{1}{\sqrt{2}\rho^2}\mathrm{e}^{\mathrm{i}\theta}\psi_z\Big(\bar{\phi}+\bar{\psi}, i(\bar{\phi}-\bar{\psi}), 1-\bar{\phi}\bar{\psi}, 1+\bar{\phi}\bar{\psi}\Big) \; . \tag{16}$$

It follows from (4),(15) that there exists a holomorphic differential $\mathrm{d}h$ such that

$$\vec{x}_z \mathrm{d}z = \Big(\phi+\psi, -\mathrm{i}(\phi-\psi), 1-\phi\psi, 1+\phi\psi\Big)\mathrm{d}h \; . \tag{17}$$

Then we get a Weierstrass representation of stationary surface $\vec{x} : M \to \mathbb{R}^4_1$:

$$\vec{x} = 2 \;\mathrm{Re}\int \Big(\phi+\psi, -\mathrm{i}(\phi-\psi), 1-\phi\psi, 1+\phi\psi\Big)\mathrm{d}h. \tag{18}$$

in terms of two meromorphic functions ϕ, ψ and a holomorphic 1-form $\mathrm{d}h = h'(z)\mathrm{d}z$.

Remark 2.1. When $\phi \equiv -\frac{1}{\bar{\psi}}$, the above formula (18) yields a minimal surface in $\mathbb{R}^3$ and we recover the classical Weierstrass representation.

When $\phi \equiv \frac{1}{\bar{\psi}}$, (18) gives the Weierstrass representation for a maximal surface in $\mathbb{R}^3_1$.

When ϕ or ψ is constant, without loss of generality (see Remark 2.3 and (19)) we may assume $\psi \equiv 0$. After integration we get $x_3 - x_4 = \textit{constant}$, hence $\vec{x}$ is a zero mean curvature surface in a 3-space $\mathbb{R}^3_0$ (which is endowed with a degenerate inner product).

Thus all these classical cases are included as special cases of our generalized Weierstrass type representation.

Definition 2.2. *Similar to the case of minimal surfaces in* $\mathbb{R}^3$*, we call* ϕ, ψ the Gauss maps *of* $\vec{x}$*, and* $\mathrm{d}h$ the height differential.

Remark 2.3. It is important to consider the effect of a Lorentz isometry of $\mathbb{R}^4_1$ on the Weierstrass data, which will be frequently used to simplify the construction of examples, and to reduce general situation to special cases. Observe that the induced action on the projective light-cone is nothing but a Möbius transformation on S^2, or just a fractional linear transformation on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ given by $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathbb{C}, \; ad - bc = 1$. The Gauss maps ϕ, ψ and the height differential $\mathrm{d}h$ transform as below:

$$\phi \Rightarrow \frac{a\phi + b}{c\phi + d} \; , \quad \psi \Rightarrow \frac{\bar{a}\psi + \bar{b}}{\bar{c}\psi + \bar{d}} \; , \quad \mathrm{d}h \Rightarrow (c\phi + d)(\bar{c}\psi + \bar{d})\mathrm{d}h \; . \tag{19}$$

Theorem 2.4. *Given holomorphic* 1*-form* $\mathrm{d}h$ *and meromorphic functions* $\phi, \psi : M \to \mathbb{C} \cup \{\infty\}$ *globally defined on a Riemann surface* M*. Suppose they satisfy the regularity condition 1),2) and period condition 3) as below:*

1) $\phi \neq \bar{\psi}$ *on* M *and their poles do not coincide;*

2) The zeros of $\mathrm{d}h$ *coincide with the poles of* ϕ *or* ψ *with the same order;*

3) Along any closed path the periods satisfy

$$\oint_\gamma \phi \mathrm{d}h = -\overline{\oint_\gamma \psi \mathrm{d}h}, \quad \mathrm{Re} \oint_\gamma \mathrm{d}h = 0 = \mathrm{Re} \oint_\gamma \phi\psi \mathrm{d}h \ . \tag{20}$$

Then (18) *defines a stationary surface* $\vec{x} : M \to \mathbb{R}^4_1$*.*

Conversely, any stationary surface $\vec{x} : M \to \mathbb{R}^4_1$ *can be represented as* (18) *in terms of such* ϕ*,* ψ *and* $\mathrm{d}h$ *over a (necessarily non-compact) Riemann surface* M*.*

We give a detailed explanation of condition 1), 2) instead of proving the theorem (which is easy and similar to the known case of minimal surfaces in $\mathbb{R}^3$). For a stationary surface constructed by (17), we find the metric

$$\mathrm{e}^{2\omega}|\mathrm{d}z|^2 = \langle \vec{x}_z, \vec{x}_{\bar{z}} \rangle |\mathrm{d}z|^2 = 2|\phi - \bar{\psi}|^2 |\mathrm{d}h|^2 \ . \tag{21}$$

Thus $\phi \neq \bar{\psi}$ when the values $\phi(z), \psi(z)$ are finite. The exceptional case is at the poles of ϕ or ψ, which is equivalent to the previous situation, since we could take any point on S^2 to be the north pole up a rotation, or equivalently, take any point to be $\infty \in \mathbb{C}P^1$ up to a linear fractional transformation as in (19). So ϕ, ψ would not have poles at the same point on M. This explains condition 1). Now condition 2) is self-evident by (21). (The geometric interpretation for $\phi \neq \bar{\psi}$ is that $[\vec{y}], [\vec{y}^*]$ are distinct at any regular point as pointed out before (11).)

In $\mathbb{R}^3$, condition 1) is satisfied automatically since we have $\phi = -\frac{1}{\bar{\psi}}$ which is never equal to $\bar{\psi}$. The corresponding lightlike normal vectors are given by $(\vec{n}, \pm 1) \in \mathbb{R}^4_1$ where $\vec{n}$ is the unit normal vector in $\mathbb{R}^3$.)

In contrast, for maximal surfaces in $\mathbb{R}^3_1$, a typical phenomenon is that they have certain singularities. The best-known example is *the catenoid in* $\mathbb{R}^3_1$ given by (18) with

$$\phi = z, \quad \psi = \frac{1}{z}, \quad \mathrm{d}h = \frac{\mathrm{d}z}{z}, \quad M = \mathbb{C} \backslash \{0\}. \tag{22}$$

After integration we obtain a rotational maximal surface $\vec{x} = (x_1, x_2, x_3, x_4)$ with

$$\sqrt{x_1^2 + x_2^2} = \sinh(x_4), \quad x_3 = 0.$$

Compared to the catenoid in $\mathbb{R}^3$, this example is peculiar in that it has a cone-like singularity, which corresponds exactly to points on the circle $|z| = 1$ where $\phi = \bar{\psi}$.

Next we express the geometrical quantities of stationary surface $\vec{x} : M \to \mathbb{R}^4_1$ in terms of the Weierstrass data ϕ, ψ and $\mathrm{d}h = \mathrm{d}h(z) = h'(z)\mathrm{d}z$. Comparing (4),(5),(15),(16), (17), and (21) we obtain

$$\Omega = \frac{1}{\sqrt{2}} \mathrm{e}^{-\mathrm{i}\theta} h'(z) \phi_z, \ \Omega^* = \frac{1}{\sqrt{2}} \mathrm{e}^{\mathrm{i}\theta} h'(z) \psi_z \ . \tag{23}$$

Then by (9) we get

$$-K + \mathrm{i}K^\perp = 4\mathrm{e}^{-4\omega} \mathrm{e}^{-2\mathrm{i}\theta} |h'(z)|^2 \phi_z \bar{\psi}_{\bar{z}} = 4\mathrm{e}^{-2\omega} \frac{\phi_z \bar{\psi}_{\bar{z}}}{(\phi - \bar{\psi})^2} = \Delta \ln(\phi - \bar{\psi}) \ . \tag{24}$$

Now we have an extremely important formula below for the total Gaussian and normal curvature over a compact stationary surface M with boundary ∂M:

$$
\begin{aligned}
\int_M (-K + \mathrm{i}K^{\perp})\mathrm{d}M = \int_M \frac{4\phi_z \bar{\psi}_{\bar{z}}}{(\phi - \bar{\psi})^2}\mathrm{d}u \wedge \mathrm{d}v = 2\mathrm{i}\int_M \frac{\phi_z \bar{\psi}_{\bar{z}}}{(\phi - \bar{\psi})^2}\mathrm{d}z \wedge \mathrm{d}\bar{z} \\
= -2\mathrm{i}\int_{\partial M} \frac{\phi_z}{\phi - \bar{\psi}}\mathrm{d}z = -2\mathrm{i}\int_{\partial M} \frac{\bar{\psi}_{\bar{z}}}{\phi - \bar{\psi}}\mathrm{d}\bar{z}.
\end{aligned} \tag{25}
$$

Remark 2.5. It is easy to see that $K^{\perp} \equiv 0$ when $\vec{x}$ is contained in a 3-dimensional subspace. By Remark 2.1 we deduce that

- $K \leq 0$ for minimal surfaces in $\mathbb{R}^3$;
- $K \geq 0$ for maximal surfaces in $\mathbb{R}^3_1$;
- $K \equiv 0$ for zero mean curvature surfaces in $\mathbb{R}^3_0$.

These agree with well-known facts. For more on curvature properties of complete stationary surfaces, see [2]. We emphasize that in general the Gaussian curvature K does not have a fixed sign. Hence the improper integral $\int_M K\mathrm{d}M$ is meaningful only when it is *absolutely convergent*.

Remark 2.6. The complex integral $\int_M (-K + \mathrm{i}K^{\perp})\mathrm{d}M$ is very important. Sometimes we say that $\vec{x} : M \to \mathbb{R}^4_1$ has *finite total curvature* if this integral converges absolutely, i.e.,

$$\int_M |-K + \mathrm{i}K^{\perp}|\mathrm{d}M < +\infty$$

This implies finite total Gaussian curvature $\int_M |K|\mathrm{d}M < +\infty$. But we do not know whether the converse is true.

Convention: In this paper, we always assume that neither of ϕ, ψ is a constant unless it is stated otherwise. According to Remark 2.1, that means we have ruled out the trivial case of stationary surfaces in $\mathbb{R}^3_0$.

3 Finite total curvature and essential singularities

Recall that a significant class of minimal surfaces in $\mathbb{R}^3$ is those complete ones with finite total Gaussian curvature, i.e.,

$$\int_M |K|\mathrm{d}M = -\int_M K\mathrm{d}M < +\infty.$$

The importance of this condition relies on the following classical result.

Theorem 3.1. *Let $(M, \mathrm{d}s^2)$ be a non-compact surface with a complete metric. Suppose $\int_M |K|\mathrm{d}M < +\infty$, then:*

(1) (Huber[14]) There is a compact Riemann surface $\overline{M}$ such that M as a Riemann surface is biholomorphic to $\overline{M}\backslash\{p_1, p_2, \cdots, p_r\}$.

(2) (Osserman[28]) When this is a minimal surface in $\mathbb{R}^3$ with the induced metric $\mathrm{d}s^2$, the Gauss map $G = \phi = -1/\psi$ and the height differential $\mathrm{d}h$ extend to each end p_j analytically.

(3) (Jorge and Meeks [15]) As in (1) and (2), suppose minimal surface $M \to \mathbb{R}^3$ has r ends and $\overline{M}$ is the compactification with genus g. The total curvature is related with these topological invariants via the Jorge-Meeks formula:

$$\int_M K\mathrm{d}M = 2\pi\left(2 - 2g - r - \sum_{j=1}^{r} d_j\right) , \tag{26}$$

Here $d_j + 1$ equals to the highest order of the pole of $\vec{x}_z\mathrm{d}z$ at p_j, and d_j is called the multiplicity at the end p_j.

Huber's conclusion (1) means *finite total curvature $\Rightarrow$ finite topology*, which is a purely intrinsic result. In particular, this is valid also for stationary surfaces in $\mathbb{R}^4_1$.

As to the extrinsic geometry of minimal $M^2 \to \mathbb{R}^3$ with finite total curvature, Osserman's result 2) shows that we have a nice control over its behavior at infinity. To our surprise, this is no longer true in $\mathbb{R}^4_1$. In particular we have counterexamples given below:

Example 3.2 ($M_{k,a}$ with essential singularities and finite total curvature)**.** *:*

$$M_{k,a} \cong \mathbb{C} - \{0\},\ \phi = z^k \mathrm{e}^{az},\ \psi = -\frac{\mathrm{e}^{az}}{z^k},\ \mathrm{d}h = \mathrm{e}^{-az}\mathrm{d}z\ . \tag{27}$$

where integer k and real number a satisfy $k \geq 2, 0 < a < \frac{\pi}{2}$.

Proposition 3.3. *Stationary surfaces $M_{k,a}$ in Examples 3.2 are regular, complete stationary surfaces with two ends at $z = 0, \infty$ satisfying the period conditions. Moreover their total curvature converges absolutely with*

$$\int_M K\mathrm{d}M = -4\pi k\ , \quad \int_M K^{\perp}\mathrm{d}M = 0\ . \tag{28}$$

Proof. Firstly, our examples have neither singular points nor singular ends. We don't need to consider $z = \infty$ where ϕ and ψ have essential singularities. The only pole of ϕ, ψ is $z = 0$ where $\phi(0) = 0 \neq \infty = \psi(0)$. Then one need only to verify $\phi \neq \bar{\psi}$ on $\mathbb{C} - \{0\}$. Suppose $\phi(z) = \overline{\psi(z)}$ for some $z \neq 0$. Using (27) and comparing the norms we see $|z| = 1$. Let $z = \mathrm{e}^{\mathrm{i}\theta}$ for some $\theta \in [0, 2\pi)$. Then the equation $\phi(z) = \overline{\psi(z)}$ is reduced to

$$\mathrm{e}^{2a\mathrm{i}\sin\theta} = -1,$$

which has no real solutions for θ when $0 < a < \frac{\pi}{2}$. This proves our first assertion.

Secondly, the period conditions are obviously satisfied, since any of

$$\mathrm{d}h = \mathrm{e}^{-az}\mathrm{d}z, \quad \phi\psi\mathrm{d}h = -\mathrm{e}^{az}\mathrm{d}z, \quad \phi\mathrm{d}h = z^k\mathrm{d}z, \quad \psi\mathrm{d}h = -z^{-k}\mathrm{d}z$$

have no residues (note that $k \geq 2$).

Thirdly, the metric of $M_{k,a}$ is complete by the following simple estimation:

$$\mathrm{d}s = |\phi - \bar{\psi}||\mathrm{d}h| = |z^k + (\bar{z})^{-k}\mathrm{e}^{a(\bar{z}-z)}||\mathrm{d}z| \sim \begin{cases} |z|^{-k}|\mathrm{d}z|, & (z \to 0); \\ |z|^{k}|\mathrm{d}z|, & (z \to \infty). \end{cases} \tag{29}$$

Finally we estimate the absolute total curvature:

$$\begin{aligned} \int_{\mathbb{C}} |-K + \mathrm{i}K^{\perp}|\mathrm{d}M &= 2\mathrm{i}\int_{\mathbb{C}} \frac{|\phi_z \bar{\psi}_{\bar{z}}|}{|\phi - \bar{\psi}|^2}\mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= 2\mathrm{i}\int_{\mathbb{C}} \frac{|az + k||k - az|}{\left||z|^{k+1} + |z|^{1-k}\mathrm{e}^{a(\bar{z}-z)}\right|^2}\mathrm{d}z \wedge \mathrm{d}\bar{z}. \end{aligned}$$

When $r = |z| \to 0$ the integrand is almost the same as $k^2 r^{2k-1}\mathrm{d}r\mathrm{d}\theta$ with respect to the polar coordinate; when $r = |z| \to \infty$ we have a similar estimation as $r^{1-2k}\mathrm{d}r\mathrm{d}\theta$. Each of these two improper integral converges when $r \to 0$ or $r \to \infty$, respectively. Thus the total curvature integral converges absolutely.

To find the exact value of the integration, we approximate $\mathbb{C}$ by domains $A_{r,R} \triangleq \{0 < r \le |z| \le R\}$. Using Stokes theorem for $A_{r,R}$ we obtain

$$\begin{aligned}\int_{A_{r,R}} (-K + \mathrm{i}K^{\perp})\mathrm{d}M &= 2\mathrm{i}\int_{A_{r,R}} \frac{\phi_z \bar{\psi}_{\bar{z}}}{(\phi-\bar{\psi})^2}\mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= 2\mathrm{i}\oint_{|z|=R} \frac{\phi_z}{\phi-\bar{\psi}}\mathrm{d}z - 2\mathrm{i}\oint_{|z|=r} \frac{\phi_z}{\phi-\bar{\psi}}\mathrm{d}z \ ,\end{aligned}$$

where

$$\frac{\phi_z}{\phi-\bar{\psi}} = \frac{a+k/z}{1+\mathrm{e}^{a(\bar{z}-z)}/|z|^{2k}} \ .$$

When $R \to \infty$ the first contour integral converges to $2\mathrm{i}\oint_{|z|=R} \frac{k}{z}\mathrm{d}z = -4\pi k$. When $r \to 0$ the second contour integral converges to 0. This completes the proof to Proposition 3.3 as well as the formula (28). $\square$

We can construct similar examples as below. The reader may compare this to the generalized catenoid in Section 7 and the generalized Enneper-type surfaces in Section 8.

Example 3.4 (Enneper surface $E_{k,a}$ with essential singularities)**.** *:*

$$E_{k,a} \cong \mathbb{C},\ \phi = z^k \mathrm{e}^{az},\ \psi = -\frac{\mathrm{e}^{az}}{z^k},\ \mathrm{d}h = z^k \mathrm{e}^{-az}\mathrm{d}z\ . \quad \Big(k \in \mathbb{Z}_{\ge 2},\ a \in (0, \frac{\pi}{2})\Big)$$

Example 3.5 (Catenoid $C_{k,a}$ with essential singularities)**.** *:*

$$C_{k,a} \cong \mathbb{C} - \{0\},\ \phi = z^k \mathrm{e}^{az},\ \psi = -\frac{\mathrm{e}^{az}}{z^k},\ \mathrm{d}h = \frac{\mathrm{e}^{-az}}{z^k}\mathrm{d}z\ . \quad \Big(k \in \mathbb{Z}_{\ge 2},\ a \in (0, \frac{\pi}{2})\Big)$$

Remark 3.6. When $a = 0$ in Example 3.4 we get *Enneper surfaces in* $\mathbb{R}^3$ *with higher dihedral symmetry*. Similarly in Example 3.5 when $a = 0, k = 1$ we get the classical catenoid in $\mathbb{R}^3$. So these examples might be regarded as deformations of Enneper surfaces and the catenoid. When $k = 1$ and $a \ne 0$, it seems that Example 3.4 and 3.5 provide new complete regular stationary surfaces in $\mathbb{R}^4_1$ with total curvature $\int_M K\mathrm{d}M = -4\pi$. Yet this understanding is wrong since the total Gaussian curvature does not converge absolutely when $k = 1$.

Remark 3.7. A theorem of Peter Li [18] says that for a complete surface with finite topological type and of quadratic area growth, under the assumption that its Gauss curvature does not change sign, it must has finite total curvature. Anyone of our examples (27) has quadratic area growth by (29) and satisfies other assumptions of his theorem except that the Gauss curvature never has a fixed sign around the end $z = \infty$. (This last assertion is left to the interested reader to verify.) So the total curvature is not necessarily finite, which is the case when $k = 1$.

In general, a minimal surface in $\mathbb{R}^3$ or a stationary surface in $\mathbb{R}^4_1$ is called an *algebraic minimal surface* if there exists a compact Riemann surface $\overline{M}$ with $M = \overline{M}\backslash\{p_1, p_2, \cdots, p_r\}$ such that $\vec{x}_z\mathrm{d}z$ is a vector valued meromorphic form defined on $\overline{M}$. In other words, the Gauss map ϕ, ψ and height differential $\mathrm{d}h$ extend to meromorphic functions/forms on $\overline{M}$. For this surface class we may establish a Gauss-Bonnet type formula (26) (see Theorem 5.6) which involves the indices of the so-called singular ends.

4 Singular ends

Let $\vec{x}: M \to \mathbb{R}^4_1$ be a complete stationary surface given by (18) with Weierstrass data $\phi, \psi, \mathrm{d}h$. Its global geometry and topology is closely related with the behavior at infinity, where we have a similar definition of *(annular) ends* like the case in $\mathbb{R}^3$. In this paper we always assume that the period condition (20) is satisfied at an annular end unless it is stated otherwise.

At an end of $\vec{x}: M \to \mathbb{R}^4_1$, although one of the four components of $\vec{x}$ tends to ∞, the surface might still be incomplete due to the Lorentz-type metric of the ambient space. The simplest example is as below:

$$\phi = z^{-2},\ \psi = z^{-3},\ \mathrm{d}h = \mathrm{d}z. \tag{30}$$

Using (18)(21), the reader can verify that the surface has an incomplete end at $z = \infty$. Notice that at this end we have $\phi(\infty) = \bar{\psi}(\infty)$ which is a typical case for such examples.

In general, although $\phi \neq \bar{\psi}$ at any regular point of a stationary $M \to \mathbb{R}^4_1$, it might happen that $\phi = \bar{\psi}$ at one of the ends. This time the total curvature (25) involves an improper integral. On the other hand, when the total curvature is finite, the worst case is that there are finite many such ends by Huber's theorem. Thus we restrict to consider isolated zeros of complex harmonic function $\phi - \bar{\psi}$ in this section.

(It is an annoying problem to solve the equation $\phi(z) - \bar{\psi}(z) = 0$ or to show non-existence of solutions for given functions ϕ, ψ over a given Riemann surface M. See Section 6 to 9 for examples and discussions.)

Below we always use D to denote a neighborhood of $z = 0$ on $\mathbb{C}$ and D is homeomorphic to a disc. By $D \to \{0\}$ we mean any limit process, or a sequence of smaller and smaller neighborhood $\cdots \supset D_i \supset D_{i+1} \supset \cdots$ whose intersection is $\{0\}$.

4.1 Two types of singular ends

Definition 4.1. *Suppose $\vec{x}: D - \{0\} \to \mathbb{R}^4_1$ is an annular end of a regular stationary surface (with boundary) whose Gauss maps ϕ and ψ extend to meromorphic functions on D (namely, $z = 0$ is a removable singularity or a pole for both ϕ, ψ). It is called a* regular end *when*

$$\phi(0) \neq \bar{\psi}(0). \quad (\textit{Thus } \phi(z) \neq \bar{\psi}(z),\ \forall\ z \in D.)$$

It is a singular end *if $\phi(0) = \bar{\psi}(0)$ where the value could be finite or ∞ (i.e., $z = 0$ is a pole of both ϕ and ψ). Such ends are divided into two classes depending on whether functions $\phi, \bar{\psi}$ take the same value at $z = 0$ with the same multiplicity or not. When the multiplicities are equal we call it* bad singular end. *Otherwise it is a* good singular end.

The total Gaussian curvature is finite around a regular end by (25). As to a singular end, we may assume that $\phi = \bar{\psi} = 0$ at $z = 0$. Without loss of generality we may write it out more explicitly:

$$\phi = z^m + o(z^m),\ \psi = bz^n + o(z^n)$$

for complex number $b \neq 0$ and positive integers m, n. Write $z = re^{\mathrm{i}\theta}$ with $r \geq 0$ and

$\theta \in [0, 2\pi)$. At a good singular end $m \neq n$. Suppose $m > n \geq 1$. Then by (25),

$$\int_D (-K + \mathrm{i}K^{\perp})\mathrm{d}M = 2\mathrm{i}\int_D \frac{\phi_z \bar{\psi}_{\bar{z}}}{(\phi - \bar{\psi})^2}\mathrm{d}z \wedge \mathrm{d}\bar{z}$$
$$= -4\int_D \frac{r^{m+n-2}(\mathrm{e}^{\mathrm{i}(m-1)\theta} + o(r))(\mathrm{e}^{-\mathrm{i}(n-1)\theta} + o(r))}{r^{2n}(b\mathrm{e}^{-\mathrm{i}(n-1)\theta} + o(r))^2} r\mathrm{d}r\mathrm{d}\theta \to 0. \quad (D \to \{0\})$$

This integral converges absolutely.

Remark 4.2. In particular, by (25) we see that the line integral

$$\oint_{\partial D} \frac{\phi_z}{\phi - \bar{\psi}}\mathrm{d}z \quad \text{and} \quad \oint_{\partial D} \frac{\bar{\psi}_{\bar{z}}}{\phi - \bar{\psi}}\mathrm{d}\bar{z}$$

both have well-defined limit when the neighborhood $D \to \{0\}$.

If this is a bad singular end, $m = n \geq 1$, then

$$\int_D (-K + \mathrm{i}K^{\perp})\mathrm{d}M = -4\int_D \left(\frac{f(\mathrm{e}^{\mathrm{i}\theta})}{r} + o\left(\frac{1}{r}\right)\right)\mathrm{d}r\mathrm{d}\theta \ .$$

This integration does not converge absolutely. Moreover it does depend on the limit process of $D \to \{0\}$. As an explicit example, one can verify that the integral

$$\int_{\partial A} \frac{\mathrm{d}z}{z - 2\bar{z}}, \qquad A = \{z|\mathrm{Re}(z) \in (-a, a), \mathrm{Im}(z) \in (-b, b)\}$$

does depend on the choice of positive real numbers a, b. Hence by (25) the curvature integral when $\phi(z) = z, \psi(z) = 2z$ is divergent around any neighborhood of $z = 0$.

In summary we have proved

Proposition 4.3. *A singular end of a stationary surface $\vec{x} : D - \{0\} \to \mathbb{R}^4_1$ is* good *if and only if the curvature integral* (25) *converges absolutely around this end. Around a regular end or a good singular end, the two line integrals $\oint_{\partial D} \frac{\phi_z}{\phi - \psi}\mathrm{d}z$ and $\oint_{\partial D} \frac{\bar{\psi}_{\bar{z}}}{\phi - \bar{\psi}}\mathrm{d}\bar{z}$ has a well-defined limit when $D \to \{0\}$.*

Later we will see that there exist complete stationary surfaces of finite total curvature with finite many good singular ends (see Example 6.8 and 6.9).

4.2 Index of a good singular end

To establish a Gauss-Bonnet type formula relating the total Gaussian curvature and the topology for such surfaces we need to define the index of a good singular end. This is equivalent to the usual index of a zero of the complex function $\phi - \bar{\psi}$ (suppose at the end $z = 0$, $\phi(0) = \bar{\psi}(0) \neq \infty$ without loss of generality).

Lemma 4.4. *Denote $D_\varepsilon = \{z||z| < \varepsilon\}$. Let m, n be non-negative integers. Then*

$$\oint_{\partial D_\varepsilon} d\ln(z^m - \bar{z}^n) = \begin{cases} 2\pi\mathrm{i} \cdot m, & m < n; \\ -2\pi\mathrm{i} \cdot n, & m > n. \end{cases} \tag{31}$$

Proof. The result is trivial when $m = 0$ or $n = 0$. If $m, n \geq 1$, we have

$$\oint_{\partial D_\varepsilon} d\ln(z^m - \bar{z}^n) = \oint_{\partial D_\varepsilon} \frac{mz^{m-1}}{z^m - \bar{z}^n}\mathrm{d}z + \oint_{\partial D_\varepsilon} \frac{-n\bar{z}^{n-1}}{z^m - \bar{z}^n}\mathrm{d}\bar{z} \ .$$

$$\text{If } m>n, \quad \lim_{\varepsilon\to 0}\oint_{\partial D_\varepsilon}\frac{mz^{m-1}}{z^m-\bar{z}^n}\mathrm{d}z=0\ , \qquad \lim_{\varepsilon\to 0}\oint_{\partial D_\varepsilon}\frac{-n\bar{z}^{n-1}}{z^m-\bar{z}^n}\mathrm{d}\bar{z}=-n\cdot 2\pi\mathrm{i}\ .$$

$$\text{If } m<n, \quad \lim_{\varepsilon\to 0}\oint_{\partial D_\varepsilon}\frac{mz^{m-1}}{z^m-\bar{z}^n}\mathrm{d}z=m\cdot 2\pi\mathrm{i}\ , \qquad \lim_{\varepsilon\to 0}\oint_{\partial D_\varepsilon}\frac{-n\bar{z}^{n-1}}{z^m-\bar{z}^n}\mathrm{d}\bar{z}=0\ .$$

Since the integral is independent to the choice of D_ϵ, taking the limit $\epsilon\to 0$ we get the result. □

Observe that according to Remark 4.2, all these integrals above has a well-defined limit when we consider generic neighborhood D_p and general limit process $D_p\to\{p\}$. Moreover we can state our results in a more general way as below and the proof is direct.

Lemma 4.5. *Suppose p is an isolated zero of $\phi-\bar{\psi}$ in p's neighborhood D_p, where holomorphic functions ϕ and ψ take the value $\phi(p)=\overline{\psi(p)}$ with multiplicity m and n, respectively. Then we have:*

$$\textit{If } m>n, \quad \lim_{D_p\to\{p\}}\oint_{\partial D_p}\frac{\phi_z}{\phi-\bar{\psi}}\mathrm{d}z=0\ , \qquad \lim_{D_p\to\{p\}}\oint_{\partial D_p}\frac{\bar{\psi}_{\bar{z}}}{\phi-\bar{\psi}}\mathrm{d}\bar{z}=-n\cdot 2\pi\mathrm{i}\ .$$

$$\textit{If } m<n, \quad \lim_{D_p\to\{p\}}\oint_{\partial D_p}\frac{\phi_z}{\phi-\bar{\psi}}\mathrm{d}z=m\cdot 2\pi\mathrm{i}\ , \qquad \lim_{D_p\to\{p\}}\oint_{\partial D_p}\frac{\bar{\psi}_{\bar{z}}}{\phi-\bar{\psi}}\mathrm{d}\bar{z}=0\ .$$

In particular,

$$\frac{1}{2\pi\mathrm{i}}\oint_{\partial D_p} d\ln(\phi-\bar{\psi})=\begin{cases} m, & m<n;\\ -n, & m>n.\end{cases} \tag{32}$$

These integer-valued topological invariants help us to define two kinds of indices at a good singular end $\vec{x}:D_p-\{p\}\to\mathbb{R}^4_1$.

Definition 4.6. The index of $\phi-\bar{\psi}$ at p *is*

$$\mathrm{ind}_p(\phi-\bar{\psi})\triangleq\lim_{D_p\to\{p\}}\frac{1}{2\pi\mathrm{i}}\oint_{\partial D_p} d\ln(\phi-\bar{\psi})\ . \tag{33}$$

The absolute index of $\phi-\bar{\psi}$ at p *is*

$$\mathrm{ind}^+_p(\phi-\bar{\psi})\triangleq\left|\mathrm{ind}_p(\phi-\bar{\psi})\right|. \tag{34}$$

Remark 4.7. For a regular end our index is still meaningful with

$$\mathrm{ind}=\mathrm{ind}^+=0.$$

So these indices distinguish regular ends from singular ends.

Remark 4.8. Note that our definition of index of $\phi-\bar{\psi}$ is invariant under the action of fractional linear transformation (19). So it is well-defined for a stationary surface with good singular ends and independent to the choice of coordinates of $\mathbb{R}^4_1$. In particular, we can always assume that our singular ends do not coincide with poles of ϕ,ψ, hence the definition above is valid.

We notice that the poles of ϕ or ψ are also singularities of $d\ln(\phi-\bar{\psi})$ in a general sense. They contribute to the indices of singularities according to

Lemma 4.9. *Suppose ϕ is a meromorphic function in a neighborhood D_q of q with one pole of order k, ψ is holomorphic in D_q. Then*

$$\oint_{\partial D_q} d\ln(\phi-\bar{\psi}) = \lim_{D_q\to\{q\}}\oint_{\partial D_q}\frac{\phi_z}{\phi-\bar{\psi}}\mathrm{d}z = -2\pi\mathrm{i}\cdot k \ ,$$
$$\lim_{D_q\to\{q\}}\oint_{\partial D_q}\frac{\bar{\psi}_{\bar{z}}}{\phi-\bar{\psi}}d\bar{z} = 0 \ .$$

(When ϕ is holomorphic and ψ has a pole we have similar result with a different sign.)

Proof. By the residue theorem, and simple estimation using

$$\frac{\phi_z}{\phi-\bar{\psi}} - \frac{\phi_z}{\phi-\overline{\psi(q)}} = \frac{(\bar{\psi}-\overline{\psi(q)})\phi_z}{(\phi-\bar{\psi})(\phi-\overline{\psi(q)})} \ .$$

$\square$

4.3 An index theorem

Proposition 4.10. *Let $\phi,\psi:\overline{M}\to\mathbb{C}\cup\{\infty\}$ be meromorphic functions on compact Riemann surface $\overline{M}$. Suppose $\phi=\bar{\psi}$ on $\{p_j, j=1,\cdots,k\}$. Denote the usual degree of a holomorphic mapping by* $\deg\phi$ *and alike. Then we have*

$$\sum{}_j \mathrm{ind}_{p_j}(\phi-\bar{\psi}) = \deg\phi - \deg\psi \ . \tag{35}$$

Proof. Suppose ϕ,ψ have distinct poles $\{q_l\}$ and $\{\hat{q}_m\}$, whose orders sum to $\deg\phi$ and $\deg\psi$, respectively. Without loss of generality, assume p_j's are distinct from them. $d\ln(\phi-\bar{\psi})$ is an exact 1-form on the supplement of these distinct points. Using Stokes formula we get

$$\begin{aligned}
2\pi\mathrm{i}\cdot\sum_{j=1}^{k}\mathrm{ind}_{p_j}(\phi-\bar{\psi}) &= \sum_{j=1}^{k}\lim_{D_{p_j}\to\{p_j\}}\oint_{\partial D_{p_j}} d\ln(\phi-\bar{\psi}) \\
&= -\sum_{l=1}^{k_1}\lim_{D_{q_l}\to\{q_l\}}\oint_{\partial D_{q_l}} d\ln(\phi-\bar{\psi}) - \sum_{m=1}^{k_2}\lim_{D_{\hat{q}_m}\to\{\hat{q}_m\}}\oint_{\partial D_{\hat{q}_m}} d\ln(\phi-\bar{\psi}) \\
&= 2\pi\mathrm{i}(\deg\phi-\deg\psi) \ .
\end{aligned}$$

The last equality follows from Lemma 4.9 (note that $\psi(q_l)$ is a complex number when q_l is a pole of ϕ). The conclusion is thus proved. $\square$

Corollary 4.11. *Let ϕ and ψ be holomorphic maps from $\overline{M}$ to $\mathbb{C}P^1$ satisfying $\phi\neq\bar{\psi}$ on $\overline{M}$. Then* $\deg\phi=\deg\psi$.

This corollary could also be proved by considering the mapping $\phi-\bar{\psi}:\overline{M}\to\mathbb{C}P^1=\mathbb{C}\cup\{\infty\}$ and counting the number of inverse images of 0 and ∞, separately. Proposition 4.10 appears later as a corollary of Theorem 5.1. See (39).

5 Gauss-Bonnet type formulas

After discussing singular ends and their indices in detail, now we are prepared to state and prove the following main result:

Theorem 5.1. *Let $\vec{x} : M \to \mathbb{R}^4_1$ be a complete stationary surface given by* (18) *in terms of $\phi, \psi, \mathrm{d}h$ which are meromorphic functions/forms defined on a compact Riemann surface $\overline{M}$ (such surfaces are called* algebraic stationary surfaces*). It may have good singular ends, but no bad singular ends. Then the total Gaussian curvature and total normal curvature are related with the indices at the ends p_j (singular or regular) by the following formulas:*

$$\int_M K^\perp \mathrm{d}M = 0 \ ; \tag{36}$$

$$\int_M K\mathrm{d}M = -4\pi \deg\phi + 2\pi \sum\nolimits_j \left(\mathrm{ind}^+_{p_j}(\phi - \bar{\psi}) + \mathrm{ind}_{p_j}(\phi - \bar{\psi})\right) \tag{37}$$

$$= -4\pi \deg\psi + 2\pi \sum\nolimits_j \left(\mathrm{ind}^+_{p_j}(\phi - \bar{\psi}) - \mathrm{ind}_{p_j}(\phi - \bar{\psi})\right) \ . \tag{38}$$

From (37)(38) *we have equivalent identities:*

$$\sum\nolimits_j \mathrm{ind}_{p_j}(\phi - \bar{\psi}) = \deg\phi - \deg\psi \ . \tag{39}$$

$$\int_M K\mathrm{d}M = -2\pi \left(\deg\phi + \deg\psi - \sum\nolimits_j \mathrm{ind}^+_{p_j}(\phi - \bar{\psi})\right) \ . \tag{40}$$

Proof. Without loss of generality, assume that the meromorphic functions ϕ on $\overline{M}$ have poles $\{p_{r+1}, \cdots, p_m\}$ which are distinct from the ends $\{p_1, \cdots, p_r\}$. Take disjoint neighborhoods of $\{p_1, \cdots, p_r, p_{r+1}, \cdots, p_m\}$ respectively and denote them as

$$\{D_1, \cdots, D_r, D_{r+1}, \cdots, D_m\}.$$

By assumption, the ends are regular ends or good singular ends, around which the curvature integral must converge absolutely (Proposition 4.3). Thus the limit

$$\lim_{D_j \to \{p_j\}} \int_{\overline{M} - \bigcup D_j} (-K + \mathrm{i}K^\perp)\mathrm{d}M$$

is well-defined and independent of the limit process $D_j \to \{p_j\}$ for all $1 \le j \le m$. Apply Stokes theorem to $\overline{M} - \bigcup_{j=1}^m D_j$ and invoke (25). We obtain

$$\begin{aligned}\int_M (-K + \mathrm{i}K^\perp)\mathrm{d}M &= 2\mathrm{i} \lim_{D_j \to \{p_j\}} \int_{\overline{M} - \bigcup D_j} \frac{\phi_z \bar{\psi}_{\bar{z}}}{(\phi - \bar{\psi})^2} \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= 2\mathrm{i} \sum_{j=1}^m \lim_{D_j \to \{p_j\}} \int_{\partial D_j} \frac{\phi_z}{\phi - \bar{\psi}} \mathrm{d}z\end{aligned}$$

The singularities of the 1-form $\frac{\phi_z}{\phi - \bar{\psi}}\mathrm{d}z$ come from either the zeros of $\phi - \bar{\psi}$ when $j \le r$, or the poles of ϕ when $j \ge r + 1$. In the first case, by Lemma 4.5 the limit for each $j = 1, \cdots, r$ is either $\mathrm{ind}(\phi - \bar{\psi})$ or zero, depending on whether the index is positive or negative. In the second case, the limit is the order of pole of ϕ by Lemma 4.9. Taking sum we get (36) and (37).

(38) is derived in a similar fashion by considering $\frac{\bar{\psi}_{\bar{z}}}{\phi - \bar{\psi}}\mathrm{d}\bar{z}$ instead of $\frac{\phi_z}{\phi - \bar{\psi}}\mathrm{d}z$ when using Stokes theorem. Taking sum or difference of (37) and (38), we obtain (40) and (39), respectively. □

Note that by (37) and (38), we immediately get

Corollary 5.2 (Quantization of total Gaussian curvature)**.** *Under the same assumptions of the theorem above, when ϕ, ψ are not constants (equivalently, when $\vec{x}$ is not a flat surface in $\mathbb{R}^3_0$), there is always*

$$\int_M K \mathrm{d}M = -4\pi k \leq -4\pi,$$

where $k \geq 1$ is a positive integer.

Remark 5.3. When there are no singular ends or branch points, all indices vanish and we obtain a simplified version of these formulas:

$$\int_M K \mathrm{d}M = -4\pi \deg \phi = -4\pi \deg \psi.$$

The reason for $\deg \phi = \deg \psi$ has been explained in the comments following Corollary 4.11. In particular, for $M \to \mathbb{R}^3$, $\phi = -1/\psi$, which is never equal to $\bar{\psi}$. Hence we get the classical result.

Remark 5.4. On the other hand, even in the codimension-2 case, under our hypothesis there is still

$$\int_M K^{\perp} \mathrm{d}M = 0.$$

This looks peculiar, since one expected to see some non-trivial items corresponding to the degree of the normal bundle. A natural explanation is as below.

First observe that we may add the *light cone at infinity* to the ambient space $\mathbb{R}^4_1$ and get a conformal compact Lorentz manifold Q^4_1 [23]. This is the same as we did in Möbius geometry where the compactification $\mathbb{R}^n \cup \{\infty\} = S^n$ is obtained via an inversion or the stereographic projection. Put it rigorously, we may use the classical construction of *light cone model*. See [23] for details of Lorentzian conformal geometry, where Q^4_1 is identified with the set of null lines in $\mathbb{R}^6_2$, which is topologically $S^3 \times S^1/\{\pm 1\}$ endowed with the Lorentz metric $g_{S^3} \oplus (-g_{S^1})$.

Next, a stationary surface satisfying the assumption of Theorem 5.1 indeed could be compactified in Q^4_1. This gives a natural realization and visualization of Huber's compactification $M = \overline{M} \backslash \{p_1 \cdots, p_r\}$. The extended map

$$\overline{M} \to Q^4_1$$

is generally not C^{∞}, but $C^{1,\alpha}$ [17] (possibly branched at those p_j located on the light cone at infinity). The non-existence of *bad singular ends* should guarantee that the normal bundle has a nice extension.

Finally, $Q^4_1 = S^3 \times S^1/\{\pm 1\}$ has a globally defined time-like vector field. Restricted to any spacelike surface $M \to Q^4_1$ and consider the projection of this vector field to its normal plane at each point. We get a time-like (non-zero) global section of the 2-dimensional normal bundle, which shows that the normal bundle is trivial.

Definition 5.5. *The multiplicity of a regular or singular end p_j for a stationary surface in $\mathbb{R}^4_1$ is defined to be*

$$\widetilde{d_j} = d_j - \mathrm{ind}^+_{p_j},$$

where $d_j + 1$ is equal to the order of the pole of the vector-valued meromorphic 1-form $\vec{x}_z \mathrm{d}z$ at p_j.

At a regular end p_j, $\mathrm{ind}^+(p_j) = 0$. So our definition of the multiplicity of an end coincides with the old one for minimal $M^2 \to \mathbb{R}^3$. As a corollary of Theorem 5.1 and Definition 5.5 we obtain the following Gauss-Bonnet type theorem.

Theorem 5.6 (Generalized Jorge-Meeks formula)**.** *Given an algebraic stationary surface $\vec{x}: M \to \mathbb{R}^4_1$ with only regular or good singular ends $\{p_1, \cdots, p_r\} = \overline{M} - M$. Let g be the genus of compact Riemann surface $\overline{M}$, r the number of ends, and d_j the multiplicity of p_j. We have*

$$\int_M K \mathrm{d}M = 2\pi \left(2 - 2g - r - \sum_{j=1}^{r} \widetilde{d}_j\right) \ , \quad \int_M K^{\perp} \mathrm{d}M = 0 \ . \tag{41}$$

Proof. Up to a Lorentz rotation (19) we may assume that ϕ, ψ do not have poles at ends $\{p_1, \cdots, p_r\}$. Thus these ends are exactly the poles of $\mathrm{d}h$. So

$$\sum \mathrm{pole}(\mathrm{d}h) = \sum_{j=1}^{r}(d_j + 1) = \sum_{j=1}^{r} \widetilde{d}_j + \sum_{j=1}^{r} \mathrm{ind}^+ + r.$$

On the other hand, zeros of $\mathrm{d}h$ should be regular points of $\vec{x} : M \to \mathbb{R}^4_1$, hence corresponds precisely to poles of ϕ or ψ with the same order. Taking sum we get

$$\sum \mathrm{zero}(\mathrm{d}h) = \deg \phi + \deg \psi.$$

By the well-known formula for the meromorphic 1-form $\mathrm{d}h$ counting its zeros and poles over a compact Riemann surface $\overline{M}$:

$$\sum \mathrm{zero}(\mathrm{d}h) - \sum \mathrm{pole}(\mathrm{d}h) = -\chi(\overline{M}) = 2g - 2.$$

Together with $\int_M K\mathrm{d}M = -2\pi \left(\deg \phi + \deg \psi - \sum_j \mathrm{ind}^+\right)$ ((40) in Theorem 5.1), the first formula is proved. The second one is just (36) in Theorem 5.1). □

Remark 5.7. It is interesting to compare with Kusner's version of Gauss-Bonnet formula involving total branching order (Lemma 1.2. in [17]). Suppose $M \to \mathbb{R}^3$ is a branched immersion with a $C^{1,\alpha}$ compactification $\overline{M} \to S^3$, then Kusner says

$$\int_M K\mathrm{d}M = 2\pi(\chi(M) - \eta(M) + \beta(M)).$$

In our terms, $\chi(M) = 2 - 2g - r$ is the Euler number of M, $\eta(M) = \sum_{j=1}^{r} d_j$ is the sum of multiplicities of the ends, and $\beta(M)$ the total branching order. To relate with our result, for a good singular end p, without loss of generality we may write a local Weierstrass representation over a punctuated neighborhood of $z = 0$:

$$\phi = z^k \phi_0, \ \psi = z^{k+l}\psi_0, \ \mathrm{d}h = z^{-n}\mathrm{d}z, \tag{42}$$

where n, k, l are integers ($l > 0$), ϕ_0, ψ_0 are non-zero holomorphic functions around $z = 0$.

If $k = 0, n > 0$, we have a regular end where $\vec{x}_z \mathrm{d}z$ has a n-th order pole, hence the multiplicity at this end is $n - 1$.

If $k > 0, n = 0$, we have a good branch point with branching order k which equals $|\mathrm{ind}(\phi - \bar{\psi})| = \mathrm{ind}^+(\phi - \bar{\psi})$. Note that $\mathrm{d}s = |\phi - \bar{\psi}||\mathrm{d}h| = (|z|^k + o(|z|^k))|\mathrm{d}z|$ in this case, which fails to hold for a singular point/end ($l = 0$).

When n, k, l are all positive, it is natural to count both contribution coming from the poles of $\mathrm{d}h$ and from the zeros (branching orders)of $\phi - \bar{\psi}$. This justifies Definition 5.5 and Theorem 5.6. (For more on Gauss-Bonnet theorem involving branch points, see [8] and references therein.)

Proposition 5.8. *Let $\vec{x} : D^2 - \{0\} \to \mathbb{R}^4_1$ be a regular or a good singular end which is further assumed to be complete at $z = 0$. Then its multiplicity satisfies $\widetilde{d} \geq 1$.*

Proof. We need only to consider a good singular end with a local Weierstrass representation (42) and positive n, k, l as above. By definition, the multiplicity of this end is $d = n - 1$, and the index of singular end is $\mathrm{ind}^+ = k$. The metric $\mathrm{d}s = |\phi - \bar{\psi}||\mathrm{d}h| = (|z|^{k-n} + o(|z|^{k-n}))|\mathrm{d}z|$ is complete around $z = 0$. This implies $n - k \geq 1$. The period condition (20) excludes the possibility of $n - k = 1$. Hence $n - k \geq 2$ and we have $\widetilde{d} = d - k = n - 1 - k \geq 1$. ∎

As a direct consequence of this proposition and Theorem 5.6, we obtain

Corollary 5.9 (The Chern-Osserman type inequality)**.** *Let $\vec{x} : M \to \mathbb{R}^4_1$ be an algebraic stationary surface without bad singular ends, $\overline{M} = M \cup \{q_1, \cdots, q_r\}$, then*

$$\int K \mathrm{d}M \leq 2\pi(\chi(M) - r) = 4\pi(1 - g - r). \tag{43}$$

6 Typical methods to construct new examples

In this section and below we will construct many new complete algebraic stationary surfaces $\vec{x} : M \to \mathbb{R}^4_1$ which are topologically punctuated 2-spheres (genus zero) with finite total curvature. To find a specific surface $\vec{x} : M \to \mathbb{R}^4_1$ with desired properties we use three different but related methods:

(1) Write out the vector-valued differential $\vec{x}_z \mathrm{d}z$ directly with prescribed poles or Laurent expansions so that it has a desired behavior locally (around an end) or globally (like being a graph over a plane).

(2) Deform the expression of $\vec{x}_z \mathrm{d}z$ or the Weierstrass data of a known minimal surface in $\mathbb{R}^3$ in a controlled way to get new examples.

(3) Using geometric conditions to determine the distribution of zeros and poles of the Gauss maps ϕ, ψ and height differential $\mathrm{d}h$, and determine the parameters involved by solving the regularity condition and the period condition.

These methods could be combined, or used separately, like in our exploration of the generalized catenoid (see Example 7.1).

Remark 6.1. We would like to emphasize that to construct a regular complete stationary surface, one usually has to find a pair of meromorphic functions ϕ, ψ on a Riemann surface M such that

$$\phi(z) = \bar{\psi}(z)$$

has no solutions. This type of equation is quite unusual to the knowledge of the authors. Most of the time we have to deal with this problem by handwork combined with experience. (See discussions in the proof to Theorem 7.2 and Theorem 8.2 for example.) Note that $M \to \mathbb{R}^3$ is a rare case where we overcome this difficulty easily since $\phi = -1/\psi$ will never equal to $\bar{\psi}$.

An other difference with the classical case is about the embedding property. In $\mathbb{R}^3$ this put strong restriction on a minimal surface. Hereafter we will show that in $\mathbb{R}^4_1$ embedded complete examples are abundant.

6.1 Method 1: Prescribing $\vec{x}_z \mathrm{d}z$ with $\langle \vec{x}_z, \vec{x}_z \rangle = 0$

Here we demonstrate this method by constructing a stationary graphs over a 2-dimensional plane in $\mathbb{R}^4_1$.

Example 6.2 (A complete graph over $\mathbb{R}^2$). *Write*

$$\vec{x}_z \mathrm{d}z = \Big(1, \sqrt{2}\mathrm{i}, \cosh(z), \sinh(z)\Big)\mathrm{d}z \tag{44}$$

which satisfies

$$\langle \vec{x}_z, \vec{x}_z \rangle = 0, \quad |\vec{x}_z|^2 = 3 + |\cosh^2(z)| - |\sinh^2(z)| \in [2,4] \subset \mathbb{R}.$$

Thus it defines a completely embedded stationary graph $\vec{x} : \mathbb{C} \to \mathbb{R}^4_1$:

$$\vec{x}(u,v) = 2\Big(u, -\sqrt{2}v, \sinh(u)\cos(v), \cosh(u)\cos(v)\Big). \quad (z = u + \mathrm{i}v,\ \ u, v \in \mathbb{R})$$

It is singly periodical (with respect to v) and not flat. So the total curvature does not converge absolutely. Related with this fact, we point out that its Weierstrass data are

$$\phi = (1 - \sqrt{2})\mathrm{e}^{-z},\ \ \psi = (1 + \sqrt{2})\mathrm{e}^{-z},\ \ \mathrm{d}h = \frac{1}{2}\mathrm{e}^{z},$$

each of them has an essential singularity at the end $z = \infty$. Also note that $\mathrm{e}^{-z} \neq 0$ *and $\phi \neq \bar{\psi}$ always holds.*

Remark 6.3. In contrast to Meeks-Rosenberg's result about the uniqueness of helicoid in $\mathbb{R}^3$ [25], this is the first new example in $\mathbb{R}^4_1$ which is also non-flat, complete, simply connected, and properly embedded. In Proposition 8.3 we will see that the Enneper surface could be deformed to avoid self-intersection in $\mathbb{R}^4_1$. Based on these examples, it seems hopeless to establish a similar uniqueness theorem in $\mathbb{R}^4_1$.

To find embedded minimal surfaces in $\mathbb{R}^3$, a basic result says that any embedded minimal end must be either a catenoid end (asymptotic to a half-catenoid) or a planar end (asymptotic to a plane); one annular end with finite total curvature is embedded if and only if the multiplicity is 1.

In $\mathbb{R}^4_1$ there is much more freedom to construct embedded ends. The example below shows that the multiplicity could be arbitrary.

Example 6.4 (A complete graph over a punctured timelike plane). *Write*

$$\vec{x}_z \mathrm{d}z = \left(z^n + \frac{1}{z^n}, -\mathrm{i}\big(z^n - \frac{1}{z^n}\big), \sqrt{3}\mathrm{i}, 1\right)\mathrm{d}z. \quad n \in \mathbb{Z}, n \geq 2. \tag{45}$$

As the previous example, this is embedded as a graph over the (x_3, x_4) plane (yet punctuated at $(0,0)$) with two ends $z = 0, \infty$. The metric is

$$\mathrm{d}s^2 = |z|^n + \frac{1}{|z|^n} + 1 \geq 3.$$

So this surface is complete and regular. When $z \to 0$, $(x_1, x_2) \to \infty, (x_3, x_4) \to (0,0)$, so $z = 0$ is a planar end with an asymptotic plane. $\vec{x}_z \mathrm{d}z$ has a pole of order n at $z = 0$, so the multiplicity of this embedded planar end is $d_0 = n - 1$. Similarly the end $z = \infty$ is embedded with multiplicity $d_\infty = n + 1$. These information verify the Jorge-Meeks formula (41). For the reader's convenience we give

$$\phi = \bar{\lambda} \cdot z^n,\ \ \psi = \bar{\lambda} \cdot \frac{1}{z^n},\ \ \mathrm{d}h = \lambda \mathrm{d}z, \quad \lambda = \frac{1 + \sqrt{3}\mathrm{i}}{2}.$$

6.2 Method 2: Deforming known examples

Example 6.5 (Alías-Palmer deformation)**.** *Given a minimal surface $\vec{x}$ in $\mathbb{R}^3$ with $\vec{x}_z \mathrm{d}z = (1-g^2, \mathrm{i}(1+g^2), 2g, 0)\omega$, Alias and Palmer [2] introduced a deformation with complex parameter a:*

$$\vec{x}_z \mathrm{d}z = \Big(1 - ag^2, \mathrm{i}(1 + ag^2), (1 + a)g, (1 - a)g\Big)\omega.$$

In terms of our Weierstrass data,

$$\phi = \frac{1}{g}, \quad \psi = -ag = a \cdot \frac{-1}{\phi}, \quad \mathrm{d}h = g\omega. \tag{46}$$

It is easy to show [2] that when the original surface in $\mathbb{R}^3$ is completely immersed without real or imaginary period along any closed path, the same is true for the deformation when a is not a negative real number. Note that $\phi \neq \bar{\psi}$ in this case.

Alias and Palmer produced a generalization of the Enneper surface [2] using (46). Yet this result has several drawbacks compared with our work.

First, they did not find more general deformations producing similar Enneper-type surfaces, which we accomplish in Section 8.

Second, they did not discuss the embedding problem. We show not only that the Alias-Palmer deformation of the Enneper surface produces a surface with two self-intersection points and an embedded end (Proposition 8.5), but also find other deformations which are globally embedded in $\mathbb{R}^4_1$.

Finally, this method does not apply to catenoid. So we need to find other methods to construct generalized catenoid in $\mathbb{R}^4_1$ (Example 7.1).

We point out that generally there are many ways to deform a known example. (The generalized catenoids and Enneper surfaces given in the next two sections could also be viewed as deformations. And they could be deformed as in Example 3.4 and 3.5.) Below is another typical way of deformation which yields interesting generalization of the catenoid and the k-noids.

Example 6.6 (The generalized Jorge-Meeks k-noid)**.** *Recall that the classical Jorge-Meeks k-noid $\vec{x} = (x_1, x_2, x_3, 0) : M \to \mathbb{R}^3$ with $k \in \mathbb{Z}^+$ has genus zero and k catenoid ends. It is not embedded when $k \geq 3$ since those catenoid ends will intersect with each other when they are extended sufficient far away. It is defined on $M = \mathbb{C}P^1 \backslash \{\lambda^j | \lambda = \mathrm{e}^{\frac{2\pi \mathrm{i}}{k}}, j = 1, \cdots, k\}$ with*

$$M = \mathbb{C}P^1 \backslash \{\omega^j | \omega^k = 1\}, \quad \phi = -1/\psi = z^{k-1}, \quad \mathrm{d}h = \frac{z^{k-1}}{(z^k - 1)^2}\mathrm{d}z\ . \tag{47}$$

Observe that it has a k-fold rotational symmetry, whose action on M is $z \to z \cdot \lambda^j$ with $\lambda = \mathrm{e}^{\frac{2\pi \mathrm{i}}{k}}, j = 1, \cdots, k$.

Given two constants $a, b \in \mathbb{C}$ such that $a^2 - b^2 = 1, |a|^2 - |b|^2 > 0$, and a, b are linearly independent over $\mathbb{R}$ (for example we may take $a = \frac{\sqrt{3}}{2}, b = \frac{\mathrm{i}}{2}$). Deform the corresponding $\vec{x}_z = (v_1, v_2, v_3, 0)$ to

$$(\hat{\vec{x}}_{a,b})_z = (v_1, v_2, av_3, bv_3)\ . \tag{48}$$

Then integration via (18) *yields the generalized k-noid $\hat{\vec{x}}_{a,b} : M \to \mathbb{R}^4_1$.*

The generalized k-noids $\hat{\vec{x}}_{a,b}$ is still conformal, yet not isometric, to the original $\vec{x} : M \to \mathbb{R}^3$. But it is still regular and complete. The total curvature is the same and finite. Because the first two components of $\vec{x}_z$ are preserved, and the original $\mathrm{d}h$ is an exact differential, both the horizontal and vertical period conditions are also satisfied. The difference is that our deformation avoid self-intersection.

Proposition 6.7. *The generalized k-noid $\hat{\vec{x}}_{a,b}$ is embedded in $\mathbb{R}^4_1$.*

Proof. Suppose there is $\hat{\vec{x}}_{a,b}(z) = \hat{\vec{x}}_{a,b}(w)$. Since $v_3 \mathrm{d}z = \mathrm{d}h = d\frac{-1}{k(z^k-1)}$ and a, b are linearly independent over $\mathbb{R}$, by comparing the third and fourth components we deduce $z^k = w^k$, so $w = z \cdot \lambda^j$ with $\lambda = \mathrm{e}^{\frac{2\pi \mathrm{i}}{k}}$.

Now we need only to consider the first two components of $\hat{\vec{x}}_{a,b}(z)$ and $\hat{\vec{x}}_{a,b}(z \cdot \lambda^j)$, which are the same as the first two components of $\vec{x}(z), \vec{x}(z \cdot \lambda^j)$. Since $\vec{x}$ has a k-fold rotational symmetry on the (x_1, x_2) plane. Thus $\hat{\vec{x}}_{a,b}(z) = \hat{\vec{x}}_{a,b}(z \cdot \lambda^j)$ if, and only if, their first two components correspond to a fixed point under this rotation. According to the description of the k-noid, this corresponds to the fixed point of $z \to z \cdot \lambda^j$, i.e. $z = 0$ or $z = \infty$. But in either case we have $w = z \cdot \lambda^j = z$. This confirms the embedding property. □

This example (as well as the generalized Enneper surface discussed in Proposition 8.5) shows that the Lopez-Ros theorem [21] no longer holds true in $\mathbb{R}^4_1$.

6.3 Method 3: Determine Weierstrass data with given zeros/poles

This is the most widely used method in constructing examples, describing the related module space or showing non-existence result under given geometric and topological conditions.

Using this methodology we construct algebraic stationary surfaces with good singular ends and simple topological type, which has $-\int_M K \mathrm{d}M$ as small as possible. The discussion of how to derive these examples is somewhat tedious, and irrelevant to other parts of this paper; so we leave these details to another paper [19] on the classification of complete stationary surfaces with $\int_M K \mathrm{d}M = -4\pi$.

Example 6.8 (Genus zero, two good singular ends and $\int_M K \mathrm{d}M = -8\pi$)**.**

$$M = \mathbb{C}\backslash\{0\},\ \phi = z^2(z^2 + a),\ \psi = \frac{z^4}{z^2 + a},\ \mathrm{d}h = \frac{z^2 + a}{z^4}\mathrm{d}z.\ \ (a \in \mathbb{C}\backslash\{0\})$$

We observe that

- *Its genus $g = 0$; the number of ends $r = 2$; and* $\deg(\psi) = \deg(\psi) = 4$.
- *At the two ends $z = 0, \infty$,* $\mathrm{ind}_0 = 2, \tilde{d}_0 = 1; \mathrm{ind}_\infty = -2, \tilde{d}_\infty = 3$.
- *Using either of* (37),(38) *or* (41) *we get $\int_M K \mathrm{d}M = -8\pi$.*

The regularity, completeness and period conditions are easy to verify, except that we need to find suitable parameter a such that $\phi \neq \bar{\psi}$ on $M = \mathbb{C}\backslash\{0\}$. Denote $w = z^2$. Then $\phi(w) = w(w + a), \psi = \frac{w^2}{w+a}$. When $w \neq 0$ we have

$$\phi(w) = \overline{\psi}(w) \quad \Leftrightarrow \quad |w + a|^2 = w^3/|w|^2.$$

So $w = r\omega^j$ for some $j \in \{0, 1, 2\}$ and $r \in \mathbb{R}\backslash\{0\}, \omega = \mathrm{e}^{2\pi \mathrm{i}/3}$. Insert this back into the equality above we get

$$\phi(w) = \overline{\psi}(w) \quad \Leftrightarrow \quad \exists\ j \in \{0, 1, 2\}, s.t.\ |r\omega^j + a|^2 = r.$$

It is not difficult to see that when a is a sufficiently large positive real number (e.g. $a > 1$) there is no (positive) real solution r, hence $\phi \neq \bar{\psi}$ always holds true.

(A standard proof is reducing $|r\omega^j + a|^2 = r$ to $r^2 - (a+1)r + a^2 = 0$ when $j = 1, 2$, and to $r^2 + (2a-1)r + a^2 = 0$ when $j = 0$. Then both discriminants of these two quadratic equations are negative when $a > 1$. But a geometric explanation and a comparison of the orders of magnitudes also suffices.)

Such an explicit example is a helpful supplement to the discussion of good singular ends in Section 4, and to the Gauss-Bonnet type formulas in Section 5. Below we provide a somewhat different example.

Example 6.9 (Genus zero, one good singular ends and $\int_M K \mathrm{d}M = -8\pi$)**.**

$$M = (\mathbb{C}\backslash\{0\}) \cup \{\infty\},\ \phi = z(z^2 + az + b),\ \psi = \frac{z^2}{z^2 + az + b},\ \mathrm{d}h = \frac{z^2 + az + b}{z^7}.$$

The good singular end $z = 0$ has ind $= 1$ and $\tilde{d} = 5$. We leave it to the interested reader to verify that it is regular, complete, without real period; in particular, the parameters $a, b \in \mathbb{C}$ could be chosen suitably so that $\phi \neq \bar{\psi}$ on $\mathbb{C}\backslash\{0\}$.

7 The generalized catenoid

To find a suitable generalization of the catenoid in $\mathbb{R}^4_1$, at the beginning we tried to find an annulus with two catenoid-type ends (using Method 1 above with Laurent series like (51)(52) below), and obtained

Example 7.1 (The generalized catenoid)**.** *This is defined over $M = \mathbb{C}\backslash\{0\}$ with*

$$\phi = z + t,\ \psi = \frac{-1}{z - t},\ \mathrm{d}h = s\frac{z - t}{z^2}\mathrm{d}z. \qquad (-1 < t < 1, s \in \mathbb{R}\backslash\{0\}) \tag{49}$$

When $t = 0$, it is the classical catenoid in $\mathbb{R}^3$. Conversely, our construction could be viewed as the most natural deformation of catenoid with real parameter t, which preserves regularity, completeness, and period condition.

In contrast, the catenoid in $\mathbb{R}^3_1$ given by (22) has singularities; and the deformation used in [2] fails to give a suitable generalization since the period condition could not be satisfied.

In view of these facts, the uniqueness theorem below is a nice characterization of the generalized catenoid.

Theorem 7.2. *A completely immersed algebraic stationary surface in $\mathbb{R}^4_1$ with total curvature $\int K = -4\pi$ and two regular ends is a generalized catenoid given above.*

Proof. For a complete and immersed algebraic stationary surface $M \to \mathbb{R}^4_1$ with $\int K = -4\pi$ and two regular ends, by Corollary (5.9) it has genus $g = 0$. By Huber's theorem M is homeomorphic to $\mathbb{C}\backslash\{0\}$ with two ends at $z = 0, \infty$.

Next, by (37) and (38), $\int K = -4\pi$ implies that the Gauss maps ϕ, ψ have degree 1, hence they are fractional linear functions on $\mathbb{C}$.

At first sight, ϕ, ψ have six coefficients to choose arbitrarily. But we can apply a Lorentz transformation in the ambient $\mathbb{R}^4_1$ (whose action on $\phi, \bar{\psi}$ are linear fractional transformations according to (19) in Remark 2.3), or a change of complex coordinate $z \to (\alpha z + \beta)/(\gamma z + \delta)$, to simplify the expressions of ϕ, ψ, which will give a congruent surface. This is what we want to do below.

Without loss of generality, we suppose $\phi(\infty) = \infty$, $\psi(\infty) = 0$. Otherwise, if $\phi(\infty) = -d/c, \psi(\infty) = -\bar{b}/\bar{a}$, we may use linear fractional transformation $\phi \to (a\phi + b)/(c\phi + d), \psi \to (\bar{a}\psi + \bar{b})/(\bar{c}\psi + \bar{d})$ (which is non-degenerate since $\phi \neq \bar{\psi}$) and the effect is as desired.

Next, up to a change of the complex coordinate $z \to \rho e^{i\theta} z$, we may normalize

$$\phi = z + t$$

(with parameter $t \in \mathbb{R}$); meanwhile, $\psi = \frac{c}{z+r}$ with parameters $c, r \in \mathbb{C}$.

Once ϕ, ψ are given as above with $r \neq 0$, the height differential $\mathrm{d}h$ must have a unique simple zero at $z = -r$ and no poles on $\mathbb{C}\backslash\{0\}$ due to the regularity condition (Theorem 2.4). Thus

$$\mathrm{d}h = s\frac{z+r}{z^k}\mathrm{d}z$$

with parameters $s \in \mathbb{C}, k \in \mathbb{Z}$. Completeness implies $1 \leq k \leq 3$. By Proposition 5.8, each end has multiplicity at least 2, hence $k = 2$. When $r = 0$ this expression for $\mathrm{d}h$ is still valid. So we have

$$\phi = z + t, \quad \psi = \frac{c}{z+r}, \quad \mathrm{d}h = s\frac{z+r}{z^2}\mathrm{d}z, \qquad t \in \mathbb{R},\ c, r, s \in \mathbb{C}.$$

To decide the possible values of the parameters, invoking the period condition (20) (which has been used to derive Proposition 5.8 and $k = 2$), we deduce

$$s, c \in \mathbb{R}\backslash\{0\}, \quad r = -t \in \mathbb{R}.$$

Now that $\phi = z + t, \psi = \frac{c}{z-t}$, by the regularity condition $\phi \neq \bar{\psi}$, the parameters $c, t \in \mathbb{R}$ must be chosen so that the equation

$$z\bar{z} + t(\bar{z} - z) - c - t^2 = 0$$

has no solution z. If there is some z, c, t satisfying this equation, comparing the imaginary and real parts separately shows that either $t = 0, c \geq 0$, or $z \in \mathbb{R}, c + t^2 \geq 0$. Thus $\phi \neq \bar{\psi}$ always holds true if, and only if, the real parameters c, t satisfy

$$c < -t^2.$$

After the following change of complex coordinate z, together with a change of frames in ambient space $\mathbb{R}^4_1$ (see Remark 2.3):

$$\tilde{z} = \frac{1}{\sqrt{-c}}z, \quad \tilde{\phi} = \frac{1}{\sqrt{-c}}\phi, \quad \tilde{\psi} = \frac{1}{\sqrt{-c}}\psi, \quad \widetilde{\mathrm{d}h} = \sqrt{-c}\,\mathrm{d}h\ ,$$

we get the desired Weierstrass data

$$\tilde{\phi} = \tilde{z} + \tilde{t}, \quad \tilde{\psi} = \frac{-1}{\tilde{z} - \tilde{t}}, \quad \widetilde{\mathrm{d}h} = \tilde{s}\frac{\tilde{z} - \tilde{t}}{\tilde{z}^2}d\tilde{z},$$

with parameters $\tilde{c} = -1, \tilde{s} = \sqrt{-c}s \in \mathbb{R}\backslash\{0\}$ and

$$\tilde{t} = \frac{t}{\sqrt{-c}} \in (-1, 1) \subset \mathbb{R}.$$

This finishes the proof. In particular, the discussion above has shown that Example 7.1 is completely immersed without singular ends. $\square$

It is interesting to examine the properties of the generalized catenoid and compare it with the catenoid in $\mathbb{R}^3$.

Proposition 7.3. *The generalized catenoid has the following properties:*

(1) It is embedded in $\mathbb{R}^4_1$.

(2) There is a symmetry between its two ends. In other words, there is an isometry of $\mathbb{R}^4_1$ *which interchanges these two ends and preserves the whole surface invariant.*

(3) Unlike the catenoid in $\mathbb{R}^3$ *or* $\mathbb{R}^3_1$*, the generalized catenoid has no rotational symmetry when* $t \neq 0$.

(4) Two generalized catenoids are congruent to each other (up to a dilation and a Lorentz isometry) if and only if their parameters t *share the same value of* $|t|$.

(5) It is not contained in any 3-dimensional subspace when $t \neq 0$*. Each end is embedded and asymptotic to a half catenoid in a 3-dimensional Euclidean subspace.*

Proof. Up to a dilation and a translation we may take $s = 1$ and write

$$\vec{x} \;=\; 2\,\mathrm{Re}\left[\left(\left(z+\frac{t^2+1}{z}\right), -\mathrm{i}\left(z+\frac{t^2-1}{z}\right), 2\ln z, \frac{2t}{z}\right)\right]\,. \tag{50}$$

To show embeddedness, it suffices to show that the four components of $\vec{x}$ determine a unique z on $\mathbb{C}\backslash\{0\}$. The third component gives the module $|z|$. Combined with this information, using the fourth and the second component of $\vec{x}$ we can derive the real and imaginary part of z, separately. Thus there is a unique z corresponding to a given $\vec{x}(z)$. So the generalized catenoid is embedded.

The end behavior is determined by the Laurent expansion

$$\vec{x}_z \;=\; \frac{1}{z^2}\vec{w}_{-2} + \frac{1}{z}\vec{w}_{-1} + \vec{w}_0, \tag{51}$$

where the coefficient vectors (written as column vectors) are

$$\vec{w}_{-2} = \begin{pmatrix} -(1+t^2) \\ -\mathrm{i}(1-t^2) \\ 0 \\ -2t \end{pmatrix}, \quad \vec{w}_{-1} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{w}_0 = \begin{pmatrix} 1 \\ -\mathrm{i} \\ 0 \\ 0 \end{pmatrix}\,. \tag{52}$$

One may verify directly that the Lorentz transformation $\vec{x} \to A_0\vec{x}$ with

$$A_0 = \frac{1}{1-t^2}\begin{pmatrix} -1-t^2 & 0 & 0 & 2t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2t & 0 & 0 & 1+t^2 \end{pmatrix}$$

will preserve the vector-valued 1-form $\vec{x}_z \mathrm{d}z$ but interchange its two poles. This proves conclusion (2).

On the other hand, if there is a 1-parameter Lorentz transformations preserving $\vec{x}_z \mathrm{d}z$ as given above, any A in this family must have invariant subspaces

$$\begin{aligned}
V_{-2} &= \mathrm{Span}\{\mathrm{Re}(\vec{w}_{-2}), \mathrm{Im}(\vec{w}_{-2})\} = \mathrm{Span}\{(\tfrac{1+t^2}{1-t^2}, 0, 0, \tfrac{2t}{1-t^2}), (0,1,0,0)\}, \\
V_{-1} &= \mathrm{Span}\{\vec{w}_{-1}\} = \mathrm{Span}\{(0,0,1,0)\}, \\
V_0 &= \mathrm{Span}\{\mathrm{Re}(\vec{w}_0), \mathrm{Im}(\vec{w}_0)\} = \mathrm{Span}\{(1,0,0,0), (0,1,0,0)\}.
\end{aligned}$$

When $t \neq 0$, V_{-2}, V_0 are distinct, and they together span a Lorentz 3-space in $\mathbb{R}^4_1$ which is orthogonal to $\vec{w}_{-1}$. From these information it is easy to see that the Lorentz transformation A must be identity. This proves (3).

We observe that for any spacelike 3-space V in $\mathbb{R}^4_1$, there is a unique timelike vector $\vec{v}$ which is future-oriented (i.e. the fourth component is positive) and $\langle \vec{v}, \vec{v} \rangle = -1, \langle \vec{v}, V \rangle = 0$.

For $V' = V_{-2} \oplus V_{-1}$, we have $\vec{v}' = (\frac{2t}{1-t^2}, 0, 0, \frac{1+t^2}{1-t^2})$.

For $V'' = V_0 \oplus V_{-1}$, we have $\vec{v}'' = (0, 0, 0, 1)$.

An algebraic invariant associated with the pair $\{V', V''\}$ is

$$\langle \vec{v}', \vec{v}'' \rangle = \frac{1+t^2}{1-t^2},$$

which corresponds to the hyperbolic angle between the two ends, or between the spacelike 3-spaces V', V'' containing the asymptotic half catenoids (see next paragraph). Thus when the values of $|t|$ are different, the corresponding generalized catenoids are not equivalent. On the other hand, if we reverse the sign of t, the mapping x will differ by a reflection according to (50). This establishes (4).

By (50) it is easy to see that $\vec{x}(z)$ is asymptotic to the catenoid

$$\vec{x} \; = \; 2 \ \mathrm{Re} \left(z + \frac{1}{z}, -\mathrm{i}\Big(z - \frac{1}{z}\Big), 2 \ln z, 0 \right) \; .$$

in V'' when $z \to \infty, \mathrm{Re}(\frac{1}{z}) \to 0$. At the end $z = 0$ one can verify in a similar way that x is asymptotic to a half catenoid in V' (or by the symmetry between the two ends). Also note that V', V'' span the full $\mathbb{R}^4_1$ when $t \neq 0$. So the generalized catenoid is not contained in any 3-dimensional subspace. This finishes the proof to the final conclusion (5). □

Remark 7.4. By the same argument in proving conclusion (3), we see that A_0 given above is the only non-trivial symmetry of the generalized catenoid.

Remark 7.5. Concerning the conclusion (4), we can show that any stationary surface of revolution (i.e. it has a continuous 1-parameter symmetry group) must be contained in a 3-dimensional subspace of $\mathbb{R}^4_1$. Here the proof to this elementary fact is omited. Regretfully we could not find this result in the literature.

As in $\mathbb{R}^3$, for a stationary surface $\vec{x} : M \to \mathbb{R}^4_1$ there is an associated family of stationary surfaces $\vec{x}_\theta : M \to \mathbb{R}^4_1$ with $(\vec{x}_\theta)_z \mathrm{d}z = \mathrm{e}^{\mathrm{i}\theta} \vec{x}_z \mathrm{d}z$. They are locally isometric to each other, yet the period condition might be violated and the topological type might be different. A typical example is as below.

Example 7.6 (The associated family of the generalized catenoid)**.** *This is represented on* $\mathbb{C}\backslash\{0\}$ *using* (18) *with*

$$\phi = z + t, \ \psi = \frac{-1}{z-t}, \ \mathrm{d}h = \lambda \frac{z-t}{z^2} \mathrm{d}z, \tag{53}$$

where the parameter λ *is complex and* $t \in (-1, 1)$ *is real as in Example 7.1. When* λ *is not a real number, the period condition is not satisfied, and the corresponding stationary surface in* $\mathbb{R}^4_1$ *is homeomorphic to the coving space* $\mathbb{C}$.

Example 7.7 (The generalized helicoid)**.** *When* λ *is purely imaginary in* (53)*, we get the generalized helicoid in* $\mathbb{R}^4_1$.

It is well-known that the classical catenoid and helicoid in $\mathbb{R}^3$ are embedded, but any other surface in their associated family is not. The same is true in $\mathbb{R}^4_1$.

Proposition 7.8. *Generalized catenoid and helicoid are embedded. Any other stationary surface in the associated family* (53) *has self-intersection points. (Indeed, on the universal covering we have a simply-connected stationary surface whose unique end is not an embedded end.)*

Proof. Given $\phi, \psi, \mathrm{d}h$ as in (53), we write out the immersion explicitly:

$$\vec{x} \;=\; 2\ \mathrm{Re}\left[\lambda\left(z+\frac{t^2+1}{z}, -\mathrm{i}\Big(z+\frac{t^2-1}{z}\Big), 2\ln z, \frac{2t}{z}\right)\right]. \tag{54}$$

Note that $\ln z$ is a multi-valued function on $\mathbb{C}\backslash\{0\}$ whose imaginary part is given by the argument of z. There are three cases to consider:

Case 1: $\lambda = a \in \mathbb{R}$ is real and non-zero. This time we get the generalized catenoid which is shown to be embedded in Proposition 7.3.

Case 2: $\lambda = \mathrm{i}b$ is purely imaginary with $b \in \mathbb{R}$.

Notice that $\mathrm{Re}(\mathrm{i}b\ln z)$ (the third component) determines the argument of z. In other words, if $\vec{x}(z)$ and $\vec{x}(z')$ share the same third component $z, z' \in \mathbb{C}\backslash\{0\}$ must be located in the same ray emanating from $z = 0$. The fourth component of $\vec{x}$ is $\mathrm{Re}(\mathrm{i}bt/z)$, from which we can fix the module $|z|$ when z is not real. Thus the imaginary and the real part of the corresponding $w = \ln z \in \mathbb{C}$ on the universal covering are both determined uniquely.

In the exceptional case when z is on the real axis, since the second component of $\vec{x}(z)$ is essentially $z - \frac{1-t^2}{z}$, a function monotonic in the real variable z when the parameter $t \in (-1, 1)$, we still conclude that $z \to \vec{x}(z) \in \mathbb{R}^4_1$ is injective. This shows that a generalized helicoid is embedded.

Case 3: $\lambda = a + \mathrm{i}b$ and $a, b \in \mathbb{R}$ are both non-zero.

Denote $z = \rho(\cos\theta + \mathrm{i}\sin\theta) \in \mathbb{C}\backslash\{0\}$, and $w = \ln z = \ln\rho + \mathrm{i}\theta \in \mathbb{C}$ is a lift to $\mathbb{C}$, the universal covering space on which the mapping $\vec{x}$ is single-valued. We are looking for self-intersection points, i.e. pairs of $\ln z \neq \ln z'$ with $\vec{x}(z) = \vec{x}(z')$.

Given the expression (50), we may obtain the values of $\mathrm{Re}(\lambda z), \mathrm{Re}(\frac{\lambda}{z})$ from the first and the fourth components of $\vec{x}(z)$, and the value of $\mathrm{Im}(\lambda z + \frac{\lambda(t^2-1)}{z})$ from the second component. Combing these information together, we can derive the value of $z + \frac{t^2-1}{z}$ from $\vec{x}(z)$. Thus $\vec{x}(z) = \vec{x}(z')$ implies $z' = z$ or $z' = \frac{t^2-1}{z}$. The case $z' = z$ whose lifts differ by $w' - w = 2n\pi$ will not give self-intersection points (by comparing the values of the third component of $\vec{x}(z)$). Thus self-intersection happens if, and only if, the following equations are satisfied simultaneously:

$$z' = \frac{t^2-1}{z}, \tag{55}$$

$$\mathrm{Re}(\lambda z) = \mathrm{Re}(\lambda z'), \tag{56}$$

$$\mathrm{Re}(\lambda \ln z) = \mathrm{Re}(\lambda \ln z'). \tag{57}$$

Substitute $\lambda = a + \mathrm{i}b, \ln z = \ln\rho + \mathrm{i}\theta, \ln z' = \ln(\frac{t^2-1}{z}) = \ln(\frac{1-t^2}{\rho}) + \mathrm{i}(2n+1)\pi - \mathrm{i}\theta)$ into (57) where $n \in \mathbb{Z}$ is an arbitrary integer. We recognize that the solutions $z = \rho e^{\mathrm{i}\theta}$ are located on a family of logarithmic spirals

$$L_n:\ 2a\ln\rho - 2b\theta = c_n \tag{58}$$

where $c_n = a\ln(1-t^2) - b(2n+1)\pi$ is a constant depending only on the values of the parameter a, b, t, n.

On the other hand, (55) and (56) imply

$$\frac{a\cos\theta}{b\sin\theta} = \frac{\rho^2+t^2-1}{\rho^2-t^2+1}. \tag{59}$$

When z on the logarithmic spiral L_n tends to infinity, $\rho \to +\infty$, and the right hand side of (59) tends to 1. In this limit process $\theta \in (-\infty, +\infty)$ increase monotonically and $\cot\theta$ oscillates between $(-\infty, +\infty)$ periodically. Thus there are infinitely many (ρ, θ) satisfying (58) and (59) simultaneously, which finishes our proof. □

8 Generalized Enneper surfaces

Example 8.1 (The generalized Enneper surface)**.** *This is given by*

$$\phi = z,\ \psi = \frac{c}{z}\ ,\ \mathrm{d}h = s\cdot z\mathrm{d}z, \tag{60}$$

or

$$\phi = z+1,\ \psi = \frac{c}{z}\ ,\ \mathrm{d}h = s\cdot z\mathrm{d}z, \tag{61}$$

with complex parameters $c, s \in \mathbb{C}\backslash\{0\}$. The completeness and period conditions are apparently satisfied. $\vec{x}$ has no singular points if and only if the parameter $c = c_1 + \mathrm{i}c_2$ is not zero or positive real numbers in the first case (60)*, and*

$$c_1 - c_2^2 + \frac{1}{4} < 0 \tag{62}$$

in the second case (61)*.*

In (60)*, when c is a negative real number we obtain the classical Enneper surface in $\mathbb{R}^3$; when c is not a real number we obtain the deformation appearing in [2].*

Theorem 8.2. *A completely immersed algebraic stationary surface in $\mathbb{R}^4_1$ with $\int K = -4\pi$ and one regular end is a generalized Enneper surface given above.*

Proof. By (37)(38) and the assumption $\int K = -4\pi$, ϕ, ψ must be meromorphic functions with degree 1. Since on tori and higher genus compact Riemann surfaces there are no such meromorphic functions with degree 1, we know the genus must be 0 and ϕ, ψ are linear fractional functions. Suppose $M = \mathbb{C}$ with an end at $z = \infty$. Then Theorem 5.6 implies that the end has multiplicity 3.

As in the proof to Theorem 7.2, without loss of generality we may suppose $\phi(\infty) = \infty,\ \psi(\infty) = 0$ (up to a Lorentz rotation in $\mathbb{R}^4_1$).

To further simplify the expressions of linear fractional functions ϕ, ψ, let us consider the pole of ψ and the zero of ϕ. If these two points coincide, we may suppose this is the point $z = 0$ (up to a linear fractional transformation of the coordinate z). Then it is easy to see that we can simplify to get (60).

Otherwise, suppose the pole of ψ is $z = 0$ and the zero of ϕ is $z = -1$. (Using linear fractional transform to change the complex coordinate z, we may map three given points to $\infty, 0, -1$ on the Riemann sphere.) Then we have

$$\phi = a(z+1),\ \psi = \frac{c}{z},\ \mathrm{d}h = s\cdot z\mathrm{d}z,\ a, c, s \in \mathbb{C}\backslash\{0\}.$$

The height differential $\mathrm{d}h$ must have this form due to the regularity condition (Theorem 2.4). Then using (19)

$$\phi \to \phi/a,\ \psi \to \psi/\bar{a},\ \mathrm{d}h \to |a|\cdot\mathrm{d}h,$$

we get the desired Weierstrass data in (61).

The period condition (20) is obviously satisfied. By the regularity condition $\phi \neq \bar{\psi}$, the parameters $c = c_1 + \mathrm{i}c_2$ must be chosen so that the equation

$$0 = z\bar{z} + \bar{z} - \bar{c} = u^2 + v^2 + u - \mathrm{i}v - c_1 + \mathrm{i}c_2$$

about $z = u + \mathrm{i}v$ has no solution. If there are some z, c satisfying this equation, comparing the imaginary and real parts separately shows that

$$v = c_2, \quad u^2 + u + c_2^2 - c_1 = 0.$$

Thus $\phi \neq \bar{\psi}$ for any $w \in \mathbb{C} \cup \{\infty\}$ if, and only if, the discriminant of the quadratic equation above (about u) is negative, i.e. $c_1 - c_2^2 + \frac{1}{4} < 0$. This is exactly (62). □

It is well-known that for minimal surfaces in $\mathbb{R}^3$ we have a similar characterization of the Enneper surface. It always intersect with itself along a curve, hence the Enneper end is not embedded. In contrast, our deformation in $\mathbb{R}^4_1$ in general has an embedded end (as demonstrated in Proposition 8.5), or could be embedded globally in the following situation.

Proposition 8.3. *The generalized Enneper surface in* (61) *is embedded when the parameter* $c < -\frac{1}{4}$ *is real and* $s \notin \mathbb{R}$.

Proof. Using (61), we write out a generalized Enneper surface explicitly:

$$\vec{x} = \mathrm{Re}\left[s\left(\frac{z^3}{3} + \frac{z^2}{2} + cz, -\mathrm{i}\Big(\frac{z^3}{3} + \frac{z^2}{2} - cz\Big), \frac{1-c}{2}z^2 - cz, \frac{1+c}{2}z^2 + cz\right)\right].$$

If there is an intersection with $\vec{x}(z_1) = \vec{x}(z_2)$ for some $z_1 \neq z_2$, by the expression above, any of the following three functions must take the same value at z_1 and z_2:

$$\mathrm{Re}(sz^2),\ \mathrm{Re}(cs\frac{z^2}{2} - csz),\ \frac{sz^3}{3} + \frac{sz^2}{2} + \bar{s}\bar{c}\bar{z}.$$

When $c \in \mathbb{R}\backslash\{0\}$, from the first two functions above we deduce that $\mathrm{Re}(sz)$ also takes the same value at z_1 and z_2.

Without loss of generality, suppose $s = \mathrm{e}^{\mathrm{i}\theta}$. By assumption $\sin\theta \neq 0$. Denote

$$w = sz = u + \mathrm{i}v, \qquad u, v \in \mathbb{R},$$

any of the following three functions of w must take the same value at $w_1 = u_1 + \mathrm{i}v_1$ and $w_2 = u_2 + \mathrm{i}v_2$:

$$\mathrm{Re}(w);\quad \mathrm{Re}(\bar{s}w^2);\quad \mathrm{Re}\left(\frac{\bar{s}^2w^3}{3} + \frac{\bar{s}w^2}{2} + \bar{c}\bar{w}\right). \tag{63}$$

By the first one, $u_1 = u_2$. Combined with the second one, we obtain a quadratic function of v

$$\cos\theta \cdot v^2 - \sin\theta \cdot 2u_1 v$$

which takes the same value for $v = v_1, v_2$. Since $w_1 \neq w_2$, we know $v_1 \neq v_2$, so $\cos\theta \neq 0$. Vieta's formula implies

$$v_1 + v_2 = 2u_1 \tan\theta. \tag{64}$$

Similarly, by the third function in (63) above and $u_1 = u_2$, we know the quadratic function of v

$$-(u_1 + \frac{\cos\theta}{2})v^2 + (c\sin 2\theta - u_1 \sin\theta)v$$

takes the same value at $v = v_1, v_2$. The coefficient $u_1 + \frac{\cos\theta}{2}$ is non-zero. (Otherwise, if $u_1 + \frac{\cos\theta}{2} = 0 = c\sin 2\theta - u_1\sin\theta$, we deduce $c = \frac{-1}{4}$; if $u_1 + \frac{\cos\theta}{2} \neq 0 = c\sin 2\theta - u_1\sin\theta$, there must be $v_1 = v_2$. In both cases we find contradiction.) Again by Vieta's formula,

$$v_1 + v_2 = \frac{c\sin 2\theta - u_1\sin\theta}{u_1 + \frac{\cos\theta}{2}}. \tag{65}$$

Combining (64) with (65), we find that $u_1 \in \mathbb{R}$ satisfies

$$u_1^2 + \cos\theta u_1 - c\cos^2\theta = 0.$$

Thus the determinant of this quadratic function of u_1 is non-negative, which contradicts $c < \frac{-1}{4}, \cos\theta \neq 0$. This finishes our proof. $\square$

Remark 8.4. In general, embeddedness and transversal self-intersection are both open properties with respect to the parameter s. Thus it is interesting to know what will happen when s tends to a real number for the generalized Enneper surfaces as in Proposition 8.3. One can verify that $\vec{x}$ intersect itself along a curve when s is a non-zero real number, which is consistent with the open property just mentioned, because such an intersection pattern is not transversal.

For (61) with other values of parameter c, s, in general it is hard to determine whether the corresponding Enneper surfaces are embedded or not. But in the special case of (60), we have a clear conclusion. We may state this result for a class of similar examples as below:

Proposition 8.5. *The simply connected Enneper-type surface $\vec{x} : \mathbb{C} \to \mathbb{R}^4_1$ with*

$$\phi = z^k,\ \psi = \frac{c}{z^k}\ ,\ \mathrm{d}h = s \cdot z^k \mathrm{d}z, \quad k \in \mathbb{Z}^+,\ c, s \in \mathbb{C}\backslash\{0\} \tag{66}$$

is regular with two self-intersection points when $c \notin \mathbb{R}$. Thus outside a compact subset it has an embedded end (of multiplicity $d = 2k+1$).

Proof. The regularity and completeness is easy to show. In particular, we know $\phi \neq \bar{\psi}$ for $z \in \mathbb{C}$ (otherwise from $\phi = z^k, \psi = c/z^k$ we deduce $c \in \mathbb{R}$, a contradiction), and the end at $z = \infty$ is not a singular end.

By (66) we obtain

$$\vec{x} = \mathrm{Re}\left[s\left(\frac{z^{2k+1}}{2k+1} + cz, -\mathrm{i}\Big(\frac{z^{2k+1}}{2k+1} - cz\Big), \frac{1-c}{k+1}z^{k+1}, \frac{1+c}{k+1}z^{k+1}\right)\right] .$$

We compare the third and the fourth components of $\vec{x}$, which are both linear combination of the real and imaginary parts of z^{k+1}. Since $c \notin \mathbb{R}$, these two combinations are linearly independent over $\mathbb{R}$. Thus when there is a self-intersection with $\vec{x}(z) = \vec{x}(w), z \neq w$, we conclude

$$z^{k+1} = w^{k+1}, \quad w = z \cdot \mathrm{e}^{\frac{2\pi\mathrm{i}}{k+1}}.$$

The first and second components of $\vec{x}(z), \vec{x}(w)$ are equal, hence $f(z) = f(w)$ where

$$f(z) = \frac{s}{2k+1} z^{2k+1} + \bar{s}\bar{c}\bar{z}.$$

So $f(z) = f(w) = f(z e^{\frac{2\pi i}{k+1}}) = e^{\frac{-2\pi i}{k+1}} f(z)$, and

$$f(z) = 0, \quad \text{i.e. } \frac{s}{2k+1} z^{2k+1} = -\bar{s}\bar{c}\bar{z}.$$

There are $2k+2$ solutions to this equation, which have the same module $\sqrt[2k]{(2k+1)|c|}$ (a fixed constant), and differ with each other by a factor $\lambda^j (0 \le j \le 2k+1)$ with $\lambda = e^{\frac{\pi\sqrt{-1}}{k+1}}$. Given one solution z_0, then $\{z_0, z_0\lambda^2, \cdots, z_0\lambda^{2k}\}$ are mapped by $\vec{x}$ to the same point in $\mathbb{R}^4_1$, which is a self-intersection of multiplicity $k+1$. For $\{z_0\lambda, z_0\lambda^3, \cdots, z_0\lambda^{2k+1}\}$ we get the other self-intersection. This finishes our proof. □

Remark 8.6. The example (66) can be viewed as a generalization of both the example (60) and the Enneper surfaces in $\mathbb{R}^3$ with higher dihedral symmetry. (But itself does not have the dihedral symmetry unless $c \in \mathbb{R}$.)

9 Open problems

The previous results on finite total Gaussian curvature and on embedded examples encourage us to consider some deeper and harder problems.

Problem 1. *Introduce suitable index for a stationary end $\vec{x} : D\backslash\{0\} \to \mathbb{R}^4_1$ whose Gauss map ϕ or ψ has an essential singularity at $z = 0$ and the integral of Gaussian curvature converges absolutely around this end. (Then we shall establish a Gauss-Bonnet type theorem for complete stationary surfaces which must involve such indices and the total Gaussian curvature.)*

We have constructed examples like (3.2) with finite total Gaussian curvature and essential singularities for the Gauss maps ϕ, ψ. It is surprising that the total Gaussian curvature $\int K \mathrm{d}M$ (as well as $\int K^{\perp} \mathrm{d}M$) are still quantized.

Indeed, according to [11], under the assumption of finite total curvature, the area of geodesic balls of radius r at any fixed point must grow at most quadratically in r. Moreover,

$$\int_M K \mathrm{d}M + \lim_{r\to\infty} \frac{2A(r)}{r^2} = 2\pi\chi(M). \tag{67}$$

This formula unifies Jorge-Meeks formula (26) (see also Theorem 5.6) and other Gauss-Bonnet type formulas.

In our opinion, it looks plausible to introduce some topological index for a wide class of essential singularities, which we desire to be simple to compute and coincide with the area growth rate above when the end has finite total curvature. Once this is done we obtain Gauss-Bonnet type theorem by (67). Regretfully it is unclear whether finite total curvature can determine the types of essential singularities of the Gauss maps ϕ, ψ at one end in a satisfying way.

Problem 2. *Can we extend Collin's theorem to $\mathbb{R}^4_1$? In other words, assume that $\vec{x} : M \to \mathbb{R}^4_1$ is a properly embedded complete stationary surface of finite topological type (i.e., M is homeomorphic to a compact surface with finite punctures), and the number of ends is at least two, does $\vec{x}$ always has finite total Gaussian curvature?*

In Section 6 to 8 we have constructed many embedded complete stationary surfaces in $\mathbb{R}^4_1$, and refuted the conclusions of the Lopez-Ros theorem and Meeks-Rosenberg theorem under the assumption of embeddedness (see the introduction and Section 6). Another deeper results on properly embedded minimal surfaces in $\mathbb{R}^3$ is

Collin's theorem [6] If $M \subset \mathbb{R}^3$ is a properly embedded minimal surface with finite topology and more than one end, then M has finite total Gaussian curvature.

It seems that embeddedness is not quite restrictive for complete stationary surfaces in the 4-dimensional Lorentz space according to our observations before. So at the beginning we thought the conclusion of Collin's theorem might also be false in $\mathbb{R}^4_1$.

The first attempt to construct counter-examples is using the stationary graph like Example 6.2 which is complete, embedded, with genus zero and one end. We modified (44) to get

$$\vec{x}_z \mathrm{d}z = (1, \sqrt{2}\mathrm{i}, \cosh(z^2 + z^{-2}), \sinh(z^2 + z^{-2}))\mathrm{d}z.$$

Obviously, $\vec{x}_z$ is still isotropic, and the term $z^2 + z^{-2}$ is to introduce a new pole at $z = 0$, at the same time avoiding new periods by putting the power to be a even number 2. By the first two components, $\vec{x} = \mathrm{Re}\int \vec{x}_z \mathrm{d}z$ is still a graph and embedded. But $\mathrm{d}s = |\vec{x}_z||\mathrm{d}z|$ and $|\vec{x}_z| = 3 + |\cosh^2(z^2 + z^{-2})|^2 - |\sinh^2(z^2 + z^{-2})|^2 \in [2, 4]$. So $\vec{x}$ is not complete at $z = 0$. Other similar constructions will not produce counter-examples as desired, since they have the same problem.

We also tried to modify the construction above in other ways:

$$\vec{x}_z \mathrm{d}z = \left(z^k - \frac{1}{z^k}, -\mathrm{i}\big(z^k + \frac{1}{z^k}\big), 2\cosh(z), 2\sinh(z) \right) \mathrm{d}z$$

Like Example 3.4 and 3.5, their gauss maps are $\phi = z^k \mathrm{e}^z$, $\psi = -\frac{\mathrm{e}^z}{z^k}$. Thus they could not be counter-examples either. Specifically, When $k \geq 2$ the total curvature is finite; when $k = 1$, although the total curvature does not converge absolutely, the period condition is not satisfied and the topological type is not finite. (Moreover we can show that it has self-intersections in $\mathbb{R}^4_1$.)

If we consider the generalized catenoid in Example 7.1 which is complete, embedded, with genus zero and two ends, and take $t = 1$, i.e.

$$\phi = z + 1,\ \psi = \frac{-1}{z - 1},\ \mathrm{d}h = \frac{z - 1}{z^2}\mathrm{d}z,$$

then this limit case is still embedded (by the same argument as Proposition 7.3) and regular. Since now we have a bad singular end $z = 0$, the total Gaussian curvature does not converge absolutely (see Section 4). It looks as if the conclusion of Collin's theorem is refuted. The similar trouble is that $\mathrm{d}s = |\vec{x}_z||\mathrm{d}z| = |\phi - \bar{\psi}||\mathrm{d}h| = |\frac{z\bar{z} + \bar{z} - z}{z\bar{z}}||\mathrm{d}z|$, and the integration of $|\vec{x}_z||\mathrm{d}z|$ along the real axis $z \in \mathbb{R}$ gives a finite number. So this singular end $z = 0$ is not complete as desired.

From the first and the third examples above we also see that the properness no longer implies completeness like in $\mathbb{R}^3$.

Because we can not find simple counter-examples, although there are so much difference compared with the case of $\mathbb{R}^3$, for stationary surfaces in $\mathbb{R}^4_1$ we conjecture that Collin's theorem might still be valid. If this is true, it will complement previous existence results and deeply influence our understanding of the embedding problem in $\mathbb{R}^4_1$.

Problem 3. *Assume that a complete stationary surface in $\mathbb{R}^4_1$ has finite total Gaussian curvature, i.e. $\int |K| \mathrm{d}M < +\infty$. Does it imply that $\int |-K + \mathrm{i}K^{\perp}| \mathrm{d}M < +\infty$?*

This problem was mentioned in Remark 2.6. In particular, if the conclusion is true, we conjecture that the integration of the normal curvature vanishes, i.e., $\int K^{\perp} \mathrm{d}M = 0$. See Remark 5.4 for related discussions.

Problem 4. *Classify complete stationary surfaces in $\mathbb{R}^4_1$ with finite total curvature $\int K \mathrm{d}M = -4\pi$.*

It is well-known that such minimal surfaces in $\mathbb{R}^3$ are catenoids and Enneper surfaces. If we assume that it is algebraic and it has no singular ends, we can get the same conclusion (Proposition 7.3 and 8.3).

How about the case of algebraic stationary surfaces with *good singular ends*? So far we can show [19] that if such surfaces exist with $\int K \mathrm{d}M = -4\pi$, then it must be $\vec{x} : \mathbb{C}\backslash\{0\} \to \mathbb{R}^4_1$ with two good singular ends at $0, \infty$, whose Weierstrass data are

$$\phi(z) = z^m(z+a),\ \psi(z) = \frac{z^{m+1}}{z+b},\ \mathrm{d}h = \frac{z+b}{z^{m+2}}\mathrm{d}z,$$

where the parameters $a, b \in \mathbb{C}\backslash\{0\}$ must satisfy $a + b = 1$ to satisfy the period condition.

But it is very difficult to decide whether there exist a singular point z (where $\phi(z) = \overline{\psi(z)}$). In other words, (when $m = 1$) whether we can choose a and $b = 1 - a$ suitably such that the equation

$$\overline{z(z+b)} = z^2/(z+a) \tag{68}$$

has no solution except the trivial root $z = 0$ (and $z = \infty$).

Thanks to the help of our colleague Professor Bican Xia with his symbolic computation software, it has been verified that there is no such a pair $\{a, b = 1 - a\}$ as desired. Encouraged by this, we are still struggling to find a straightforward proof to this fact. Once this is achieved, we know that complete algebraic stationary surfaces with $\int K \mathrm{d}M = -4\pi$ are still generalized catenoids and Enneper surfaces.

We also observe that Example 3.2, the simplest example with essential singularities in ϕ, ψ with finite total curvature has $\int K \mathrm{d}M = -8\pi$ when $k = 2$, and $\int K \mathrm{d}M = -4\pi$ could not be realized in that family. So there might be no new examples even without the assumption of *algebraic*. Proving this conjecture is surely a much harder task.

When the total curvature $\int K \mathrm{d}M \geq -12\pi$, Meeks [24], Lopez [22] and Costa [7] have finished such classifications in $\mathbb{R}^3$. But in view of so many new examples of total curvature -8π with good singular ends (Example 6.8, 6.9), or with essential singularities for ϕ, ψ (like Example 3.2), and noticing the already existing difficulties when $\int K \mathrm{d}M = -4\pi$, it seems no hope to obtain a similar classification.

Problem 5. *Estimate the possible exceptional values of the Gauss maps ϕ and ψ of complete stationary surfaces in $\mathbb{R}^4_1$. In particular, can we generalize Fujimoto's Theorem [10]to $\mathbb{R}^4_1$?*

When the complete stationary surface is assumed to be algebraic, we can use the same method as Osserman [28] to show that if the surface has no singular ends, then the exceptional values of either of ϕ, ψ is no more than 4 [20]. If we allow singular ends, then we can obtain a weaker conclusion that at least one of the exceptional

values of any of ϕ and ψ is no more than 4. If we add the condition of finite total Gaussian curvature, then the number 4 could be replaced by 3. All these are all similar to the case of $\mathbb{R}^3$. The next step is to assume only completeness and try to generalize Fujimoto's Theorem [10].

Note that it is also interesting to consider the upper bound of the exceptional spacelike or timelike directions [20].

Problem 6. *Apply our theory to study spacelike Willmore surfaces in Lorentzian space forms and Laguerre minimal surfaces in* $\mathbb{R}^3$.

As mentioned in the introduction, this is the original motivation for our exploration reported in this paper. Stationary surfaces in $\mathbb{R}^4_1$ are special examples of spacelike Willmore surfaces in Lorentzian space forms [23]. When being complete with planar ends (the definition is similar to the case of $\mathbb{R}^3$), such surfaces will compactify to be compact Willmore surfaces in $\mathbb{Q}^4_1$, the universal compactification of 4-dimensional Lorentz space forms. This construction yields all compact Willmore 2-spheres in $\mathbb{Q}^4_1$ [23], [32]. In particular, it is interesting to know whether there exist Willmore 2-spheres in $\mathbb{Q}^4_1$ with Willmore functional $4\pi k, (k = 2, 3, 5, 7)$. Note that $k = 2, 3, 5, 7$ are all exceptional values for immersed Willmore 2-spheres in $\mathbb{S}^3$ [3], [4], [12].

A stationary surface in $\mathbb{R}^4_1$ also corresponds to the so-called Laguerre Gauss map of a Laguerre minimal surface in $\mathbb{R}^3$ [31]. So our theory provide a direct method to construct examples of Laguerre minimal surfaces and to study their global geometry.

Problem 7. *Can we have any general result about the equation*

$$\phi = \bar{\psi} \qquad\qquad (*)$$

for a pair of meromorphic functions ϕ, ψ *over a Riemann surface?*

To the best of our knowledge, there is no general theory on such equations. We collect some known facts on this problem.

- There exist many pairs of functions $\{\phi, \psi\}$ for which there is no solutions to equation $(*)$. For example, in Example 6.5, 7.1 and 8.1 we take $\psi \equiv \frac{c}{\phi+a}$. When the parameter a, c are chosen suitably there is no solution to $\frac{c}{\phi+a} = \psi = \bar{\phi}$. Another example is Example 3.2.

- For meromorphic functions ϕ, ψ over compact Riemann surface M, if there is no solution to equation $(*)$, then $\deg\phi = \deg\psi$ (see Corollary 4.11 or Theorem 5.1). This a necessary but not sufficient condition for the non-existence of solutions.

- Regard $(*)$ as zeros of complex harmonic function $\phi - \bar{\psi}$. Since this is a complex-valued function with convergent power series (analytic function), its zero locus is a union of isolated points and some analytic arcs which might meet at some vertex (at each vertex there are finitely many of such arcs).

In two concrete problems we need to deal with this technical trouble.

One is in Problem 4, where we have to find suitable a, b satisfying $a + b = 1$ so that the equation (68) has only trivial solution $z = 0$. Since the parameters a, b has one degree of freedom to choose, the discussion is quite involved. See Problem 4 where we mentioned our progress.

The other one is in Problem 1. When considering one end with an essential singularity of given ϕ, ψ , we want to know whether there will be infinitely many singular points p such that $\phi(p) = \overline{\psi(p)}$ in the neighborhood of this end. By the theorem of Weierstrass, around an essential singularity of a holomorphic function ϕ, it may take almost every complex value infinitely many times. Thus the conclusion to our problem seems quite unclear.

Final remarks

We may compare to the theory of minimal surfaces in $\mathbb{R}^4$. In that case, we still have a pair of Gauss maps ϕ, ψ into $\mathbb{C}P^1 \times \mathbb{C}P^1$. This target space is endowed with its standard Kähler form, and the unitary group action induced from $\mathbb{C}P^3$. So it suffices to study the Kähler geometry of $\mathbb{C}P^1 \times \mathbb{C}P^1$. In particular, Osserman's theorem [27] and Fujimoto's theory [10] are based on this observation.

For stationary surfaces in $\mathbb{R}^4_1$, the two Gauss maps together gives a mapping

$$\begin{array}{rccl} (\phi, \bar{\psi}): & M & \longrightarrow & Q^2 \triangleq \{(z, \bar{w}) \in \mathbb{C}P^1 \times \mathbb{C}P^1 \mid z \neq \bar{w}\}, \\ & p & \mapsto & (\phi(p), \overline{\psi(p)}). \end{array}$$

The target space is an open 2-dimensional complex manifold homeomorphic to $S^2 \times S^2$ with the diagonal removed. The projective action of the Lorentz isometries on $\mathbb{R}^4_1$ induces an action of $SL(2, \mathbb{C})$ on this Q^2 (Remark 2.3):

$$A \cdot (z, \bar{w}) \quad \mapsto \quad \left(\frac{az + b}{cz + d}, \ \frac{a\bar{w} + b}{c\bar{w} + d} \right), \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}).$$

Under this action we have an invariant complex 2-form

$$\Theta \triangleq \frac{1}{(z - \bar{w})^2} \, dz \wedge d\bar{w}, \tag{69}$$

whose pull-back to M via the mapping $(\phi, \bar{\psi})$ is exactly the curvature form $-K + \mathrm{i}K^{\perp}$ up to a constant by (25). Thus we have to study a new geometry of

$$\left(Q^2, \ \Theta, \ SL(2, \mathbb{C})\right),$$

which is related with both the problem of finite total curvature (Problem 1, 2, 3) and the value distribution problem of ϕ, ψ (Problem 5, 7).

Now let us explain several previous results from this viewpoint. It is easy to see that there is no $SL(2, \mathbb{C})$-invariant area form on any component of $\mathbb{C}P^1 \times \mathbb{C}P^1$. Thus Osserman's original argument (that finite total Gaussian curvature implies no essential singularity of the Gauss map) does not apply at here. In particular we have counter-examples in Section 3.

Singular points/ends now appear as intersections of the mapping $(\phi, \bar{\psi})$ with the diagonal $\{(z, \bar{w}) \in \mathbb{C}P^1 \times \mathbb{C}P^1 \mid z = \bar{w}\}$, on which Θ tends to ∞. Thus at singular ends the curvature integral is improper. It is a subtle question why this integral converges absolutely exactly when this is a good singular end. See Proposition 4.3.

References

[1] Alías, L. J., Palmer, B. *Deformations of stationary surfaces*, Classical and Quantum Gravity 14 (1997), no. 8, 2107-2111.

[2] Alías, L. J., Palmer, B. *Curvature properties of zero mean curvature surfaces in four-dimensional Lorentzian space forms,* Math. Proc. Camb. Phil. Soc. 124 (1998), 315-327.

[3] Bryant, R. *A duality theorem for Willmore surfaces* J. Diff. Geom. 20 (1984), 23-53.

[4] Bryant, R. *Surfaces in conformal geometry*, Proc. Sympos. Pure Math., vol. 48 (1988), pp. 227-240, Amer. Math. Soc., Providence.

[5] Chern, S. S., Osserman, R. *Complete minimal surfaces in euclidean n-space*, J. Analyse Math. 19 (1967), 15-34.

[6] Collin, P. *Topologie et courbure des surfaces minimals de* $\mathbb{R}^3$, Ann. of Math. (2) 145 (1997), 1-31.

[7] Costa, C. J. *Classification of complete minimal surfaces in R^3 with total curvature* 12π, Invent. math. 105 (1991), 273-303.

[8] Eschenburg, J. and Tribuzy, R. *Branch points of conformal mappings of surfaces*, Math. Ann. 279 (1988), 621-633.

[9] Estudillo, F. and Romero, A. , *On maximal surfaces in the n-dimensional Lorentz-Minkowski space*, Geometriae Dedicata 38 (1991), 167-174.

[10] Fujimoto, H. *On the number of exceptional values of the Gauss maps of minimal surfaces*, J. Math. Soc. Japan 40 (1988), no. 2, 235-247.

[11] Hartman, P. *Geodesic parallel coordinates in the large,* Amer. J. Math. 86 (1964), 705-727.

[12] Heller, S. *On the Classification of Willmore Spheres*, Diploma thesis, TU Berlin, 2004.

[13] Hoffman, D., Osserman, R. *The geometry of the generalized Gauss map,* Mem. Amer. Math. Soc. 28 (1980), no. 236.

[14] Huber, Alfred *On subharmonic functions and differential geometry in the large,* Comment. Math. Helv. 32 (1957), 13-72.

[15] Jorge, L. P. M., Meeks, III W. H. *The topology of complete minimal surfaces of finite total gaussian curvature,* Topology 22 (1983), 203-221.

[16] Kobayashi, O. *Maximal surfaces with conelike singularities*, J. Math. Soc. Japan 36 (1984), no. 4, 609-617.

[17] Kusner, R. *Conformal geometry and complete minimal surfaces*, Bulletin Amer. Math. Soc. 17 (1987), no. 2, 291-295.

[18] Li, P. *Complete surfaces of at most quadratic area growth*, Comment Math. Helv. 72 (1997), 67-71.

[19] Liu, Z., Ma, X., Wang, P., *Complete stationary surfaces in 4-dimensional Lorentz space with total Gaussian curvature* -4π, in preparation.

[20] Liu, Z., Ma, X., Wang, P., *The exceptional value problem of the Gauss maps of complete stationary surfaces in 4-dimensional Lorentz space*, in preparation.

[21] López, F., Ros, A. *On embedded complete minimal surfaces of genus zero*, J. Differential Geom. 33 (1991), no. 1, 293-300.

[22] López, F. *The classification of complete minimal surfaces with total curvature greater than* -12π, Trans. Amer. Math. Soc. 334 (1992), 49-74.

[23] Ma, X., Wang, P. *Spacelike Willmore surfaces in 4-dimensional Lorentzian space forms*, Sci. in China: Ser. A, Math. Vol. 51 No. 9 (2008),1561-1576.

[24] Meeks, III W. H. *The classification of complete minimal surfaces in* R^3 *with total curvature greater than* -8π, Duke Math. J. 48 (1981), no. 3, 523-535.

[25] Meeks, III W. H., Rosenberg, H. *The uniqueness of the helicoid*, Ann. of Math. (2) 161 (2005), no. 2, 727-758.

[26] Meeks, III W. H., Pérez., J. *The classical theory of minimal surfaces*, Bull. Amer. Math. Soc. 48 (2011), 325-407.

[27] Osserman, R. *Global properties of minimal surfaces in* E^3 *and* E^n, Ann. of Math. (2) 80, 1964, 340-364.

[28] Osserman, R. *A survey of minimal surfaces,* Second edition. Dover Publications, Inc., New York, 1986.

[29] Palmer, B. *Area minimization among marginally trapped surfaces in LorentzCMinkowski space*, Calc. Var. 41 (2011), no. 3, 387-395

[30] Umehara, M., Yamada, K. *Maximal surfaces with singularities in Minkowski space*, Hokkaido Math. J. 35 (2006), no. 1, 13-40.

[31] Wang, C. P. *Weierstrass representations of Laguerre minimal surfaces in* $\mathbb{R}^3$, Results Math. 52 (2008), no. 3-4, 399-408.

[32] Wang, P. *Spacelike S-Willmore spheres in Lorentzian space forms*, Pacific Journal of Mathematics 246 (2010), no. 2, 495-510.

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